Repelled point processes with application to numerical integration

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LABORATOIRE DE PROBABILITÉS STATISTIQUE & MODÉLISATION

SORBONNE UNIVERSITÉ

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Numerical integration

Let f be a continuous function supported on a compact $K \subset \mathbb{R}^d$.

- **Need:** approximate $\int_K f(\mathbf{z}) \, \mathrm{d}\mathbf{z}$.

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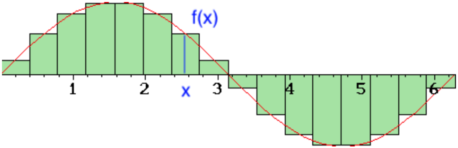

Numerical integration

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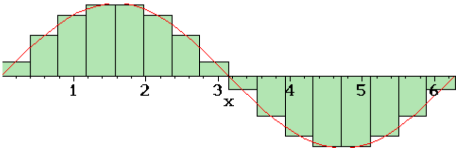
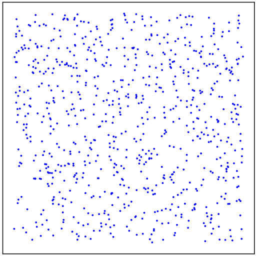


Numerical integration

- For any $\{\mathbf{z}_i\}_{i=1}^N \subset K$ and $\{w_i\}_{i=1}^N \subset \mathbb{R}$, there exists $f \in \mathcal{F}^k$ s.t.

$$\Big| \int_K f(\mathbf{z})\, \mathrm{d}\mathbf{z} - \sum_{i=1}^N w_i f(\mathbf{z}_i) \Big| \geq \frac{C_1}{N^{k/d}}.$$



Fixed $\{\mathbf{z}_i\}_{i=1}^N$

N. Bakhvalov. *Vestnik MGU, Ser. Math. Mech. Astron. Phys. Chem.*, 1959.
N. Bakhvalov. *USSR Comp. Math. and Mathematical Physics*, 1971.

Numerical integration

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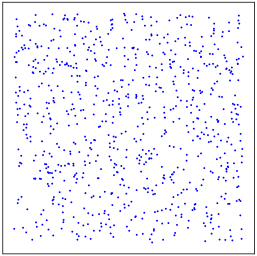
$$\Big| \int_K f(\mathbf{z})\,\mathrm{d}\mathbf{z} - \sum_{i=1}^N w_i f(\mathbf{z}_i) \Big| \geq \frac{C_1}{N^{k/d}}.$$

- For $\{\mathbf{z}_i\}_{i=1}^N$ random elements of K and $\{w_i\}_{i=1}^N \subset \mathbb{R}$, there exists $f \in \mathcal{F}^k$ s.t.

$$\mathbb{E}\Big[\Big| \int_K f(\mathbf{z})\,\mathrm{d}\mathbf{z} - \sum_{i=1}^N w_i f(\mathbf{z}_i) \Big|\Big] \geq \frac{C_2}{N^{k/d+1/2}}.$$



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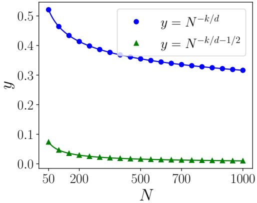


$k = 1$ and $d = 6$

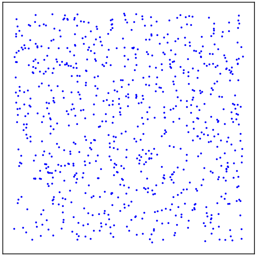
N. Bakhvalov. *Vestnik MGU, Ser. Math. Mech. Astron. Phys. Chem.*, 1959.

N. Bakhvalov. *USSR Comp. Math. and Mathematical Physics*, 1971.

Monte Carlo method

Let $\mathcal{X}$ be a stationary point process of intensity ρ

$$\int_K f(\mathbf{z})\,\mathrm{d}\mathbf{z} = \mathbb{E}\left[\sum_{\mathbf{z}\in\mathcal{X}\cap K} \rho^{-1} f(\mathbf{z})\right].$$



Samples from $\mathcal{X}$

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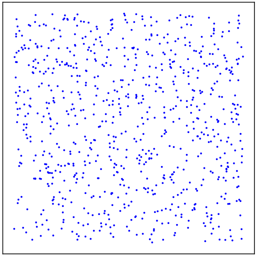
- Monte Carlo method: $\widehat{I}_{\mathcal{X}}(f) = \sum_{\mathbf{z}\in\mathcal{X}\cap K} \rho^{-1} f(\mathbf{z})$.



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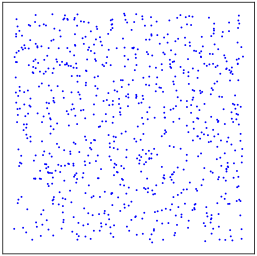
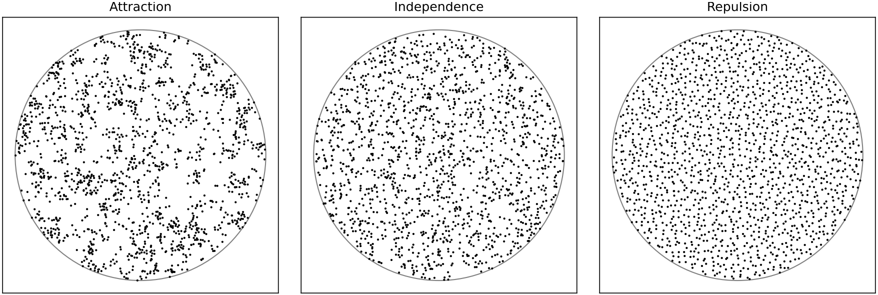
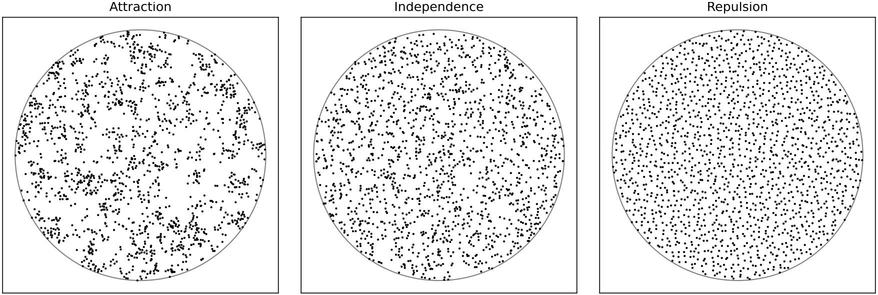


Samples from $\mathcal{X}$

- Monte Carlo method: $\widehat{I}_{\mathcal{X}}(f) = \sum_{\mathbf{z}\in\mathcal{X}\cap K} \rho^{-1} f(\mathbf{z})$.
- Number of points: $\mathcal{X}(K)$ (random).
- $N := \mathbb{E}[\mathcal{X}(K)] = \rho|K|$.

$$\widehat{I}_{\mathcal{X}}(f) = \sum_{\mathbf{z}\in\mathcal{X}\cap K} \rho^{-1} f(\mathbf{z}) \approx \int_K f(\mathbf{z})\ \mathrm{d}\mathbf{z}$$

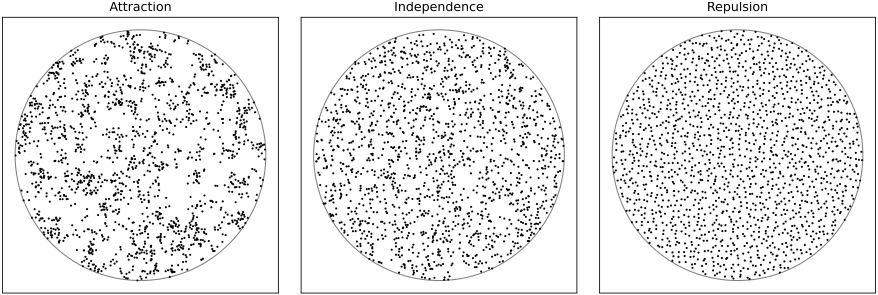




Monte Carlo with a homogeneous Poisson point process (PPP):

- Sampling from a PPP is fast.
- $\mathbb{V}\mathrm{ar}[\widehat{I}_{\mathcal{X}}(f)]^{1/2} = c(d, f)N^{-1/2}$.

A. B. Owen. *Online book*, 2013.

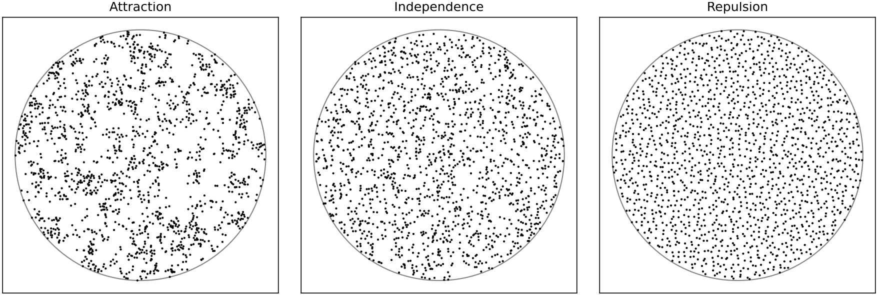


Monte Carlo with a determinantal point process (DPP) :

- $\mathbb{V}\mathrm{ar}[\widehat{I}_{\mathcal{X}}(f)]^{1/2} = O(N^{-1/2-1/(2d)})$.
- Sampling from DPPs is expensive.

R. Bardenet and A. Hardy. *The Annals of Applied Probability*, 2020.

J.-F. Coeurjolly, A. Mazoyer, and P.-O. Amblard. *Electronic Journal of Statistics*, 2021.



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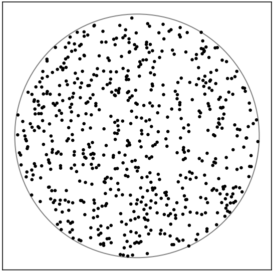
G. Gautier, R. Bardenet, and M. Valko. *Adv. in Neural Info. Processing Systems*, 2019.

Repelled point processes:

"D. Hawat, R. Bardenet, and R. Lachièze-Rey. Repelled point processes with application to numerical integration. Preprint, 2023."

Construction

$\mathcal{X}$ a stationary point process of intensity ρ of $\mathbb{R}^d$.



Sample

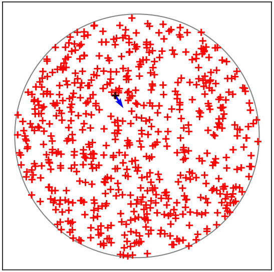
D. Hawat, R. Bardenet, and R. Lachièze-Rey. *Preprint*, 2023.

Construction

$\mathcal{X}$ a stationary point process of intensity ρ of $\mathbb{R}^d$.

- **Force:**

$$F_{\mathcal{X}}(\mathbf{a}) := \sum_{\substack{\mathbf{z} \in \mathcal{X} \setminus \{\mathbf{a}\} \\ \|\mathbf{a}-\mathbf{z}\|_2 \uparrow}} \frac{\mathbf{a}-\mathbf{z}}{\|\mathbf{a}-\mathbf{z}\|_2^d}.$$



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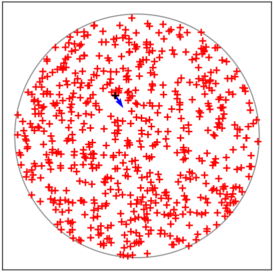
- **Repulsion operator:**

$$\Pi_\varepsilon : \mathcal{X} \longmapsto \{\mathbf{a} + \varepsilon F_{\mathcal{X}}(\mathbf{a}) : \mathbf{a} \in \mathcal{X}\}.$$



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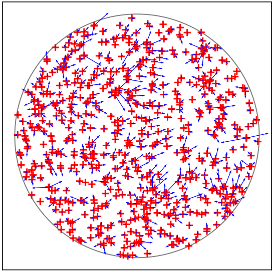

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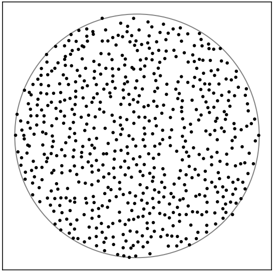

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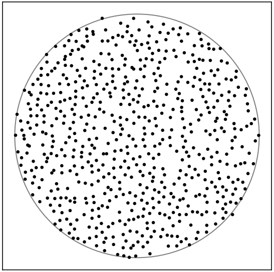
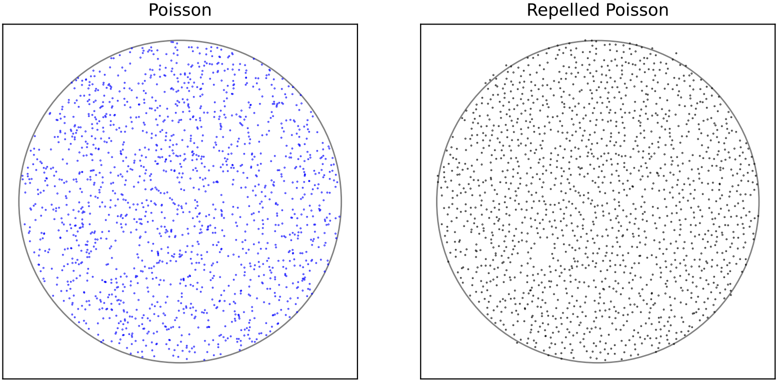


Repelled sample

D. Hawat, R. Bardenet, and R. Lachièze-Rey. *Preprint,* 2023.

Chatterjee, Sourav, Ron Peled, Yuval Peres, and Dan Romik. *Ann. of Mathematics,* 2010.

Example



D. Hawat, R. Bardenet, and R. Lachièze-Rey. *Preprint*, 2023.

Main theoretical results

Let $\mathcal{P} \in \mathbb{R}^d$ be a PPP of intensity $\rho > 0$, $d \geq 3$, and $\varepsilon \in \mathbb{R}$.

- $\Pi_\varepsilon \mathcal{P}$ is a simple, motion-invariant point process of intensity ρ.

Proposition

For any two distinct points $\mathbf{x}, \mathbf{y}$ of $\mathbb{R}^d$, the random vector $F_{\mathcal{P}}(\mathbf{x}) - F_{\mathcal{P}}(\mathbf{y})$ is continuous, i.e., for any $\mathbf{c} \in \mathbb{R}^d$,

$$\mathbb{P}\left(F_{\mathcal{P}}(\mathbf{x}) - F_{\mathcal{P}}(\mathbf{y}) = \mathbf{c}\right) = 0.$$

Moreover, $\Pi_\varepsilon \mathcal{P}$ is a stationary and isotropic point process of intensity ρ.

D. Hawat, R. Bardenet, and R. Lachièze-Rey. *Preprint*, 2023.

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- $\Pi_\varepsilon \mathcal{P}$ is a simple, motion-invariant point process of intensity ρ.
- For any $\varepsilon \in (-1, 1)$, the moments of $\Pi_\varepsilon \mathcal{P}$ exist.

Proposition

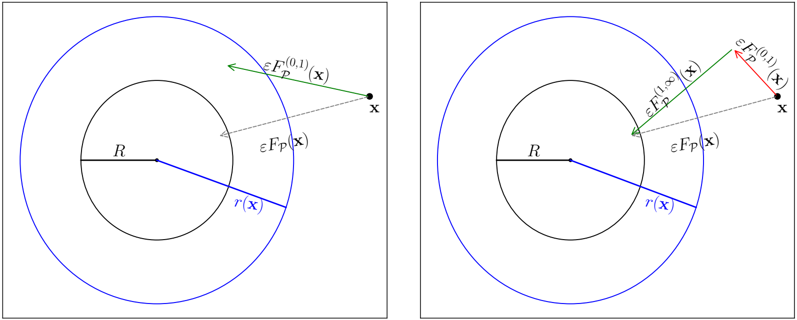
For any positive integer m and $R > 0$, we have

$$\mathbb{E}\left[\left(\sum_{\mathbf{z} \in \Pi_\varepsilon \mathcal{P}} \mathbb{1}_{B(\mathbf{0},R)}(\mathbf{z})\right)^m\right] < \infty.$$

D. Hawat, R. Bardenet, and R. Lachièze-Rey. *Preprint*, 2023.

Proof's main idea: $\mathbb{E}[(\sum_{\mathbf{z}\in\Pi_\varepsilon\mathcal{P}} \mathbb{1}_{B(\mathbf{0},R)}(\mathbf{z}))^m] < \infty$

- $F_{\mathcal{P}}(\mathbf{x}) = F_{\mathcal{P}}^{(0,1)}(\mathbf{x}) + F_{\mathcal{P}}^{(1,\infty)}(\mathbf{x})$.



(1) (2)

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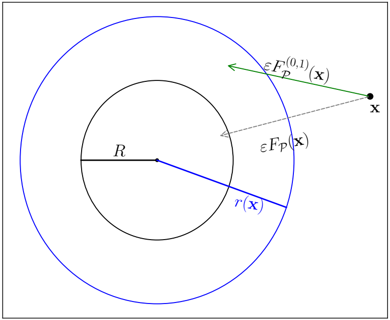
- $F_{\mathcal{P}}(\mathbf{x}) = F_{\mathcal{P}}^{(0,1)}(\mathbf{x}) + F_{\mathcal{P}}^{(1,\infty)}(\mathbf{x})$.
- For $\mathbf{x} \in \mathcal{P} \cap B(\mathbf{0}, R)^c$, if $\mathbf{x} + \varepsilon F_{\mathcal{P}}(\mathbf{x}) \in B(\mathbf{0}, R)$:
  1. $\mathbf{x} + \varepsilon F_{\mathcal{P}}^{(0,1)}(\mathbf{x}) \in B(\mathbf{0}, r(\mathbf{x}))$.
  2. $\|F_{\mathcal{P}}^{(1,\infty)}(\mathbf{x})\|_2 \geq \frac{r(\mathbf{x})-R}{|\varepsilon|}$.



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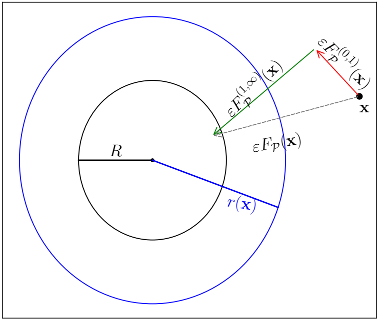

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- $\Pi_\varepsilon \mathcal{P}$ is a simple, motion-invariant point process of intensity ρ.
- For any $\varepsilon \in (-1, 1)$, the moments of $\Pi_\varepsilon \mathcal{P}$ exist.
- For $\varepsilon > 0$ small enough and $f \in C^2(\mathbb{R}^d)$, $\mathbb{V}\mathrm{ar}[\widehat{I}_{\Pi_\varepsilon \mathcal{P}}(f)] < \mathbb{V}\mathrm{ar}[\widehat{I}_{\mathcal{P}}(f)]$.

Theorem

For any function $f \in C^2(\mathbb{R}^d)$ of compact support K, we have

$$\mathbb{V}\mathrm{ar}\left[\widehat{I}_{\Pi_\varepsilon \mathcal{P}}(f)\right] = \mathbb{V}\mathrm{ar}\left[\widehat{I}_{\mathcal{P}}(f)\right](1 - 2d\kappa_d \rho \varepsilon) + O(\varepsilon^2),$$

where κ_d is the volume of the unit ball of $\mathbb{R}^d$.

D. Hawat, R. Bardenet, and R. Lachièze-Rey. *Preprint*, 2023.

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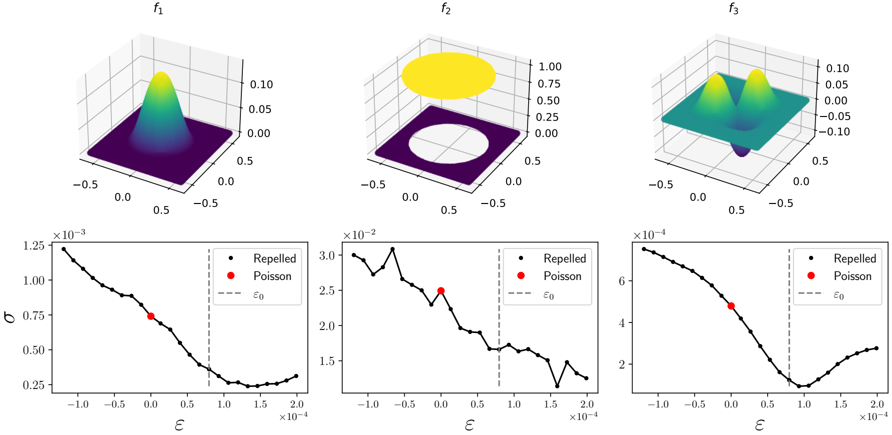
- $$F_{\mathcal{X}}(\mathbf{a}) := \sum_{\substack{\mathbf{z} \in \mathcal{X} \setminus \{\mathbf{a}\} \\ \|\mathbf{a}-\mathbf{z}\|_2 \uparrow}} \frac{\mathbf{a} - \mathbf{z}}{\|\mathbf{a} - \mathbf{z}\|_2^d}.$$
- For any $f \in C^2(\mathbb{R}^d)$ we have
  $$\int_{\mathbb{R}^d} \nabla f(\mathbf{y}) \cdot \frac{\mathbf{x} - \mathbf{y}}{\|\mathbf{x} - \mathbf{y}\|_2^d} \mathrm{d}\mathbf{y} = d\kappa_d f(\mathbf{x}).$$

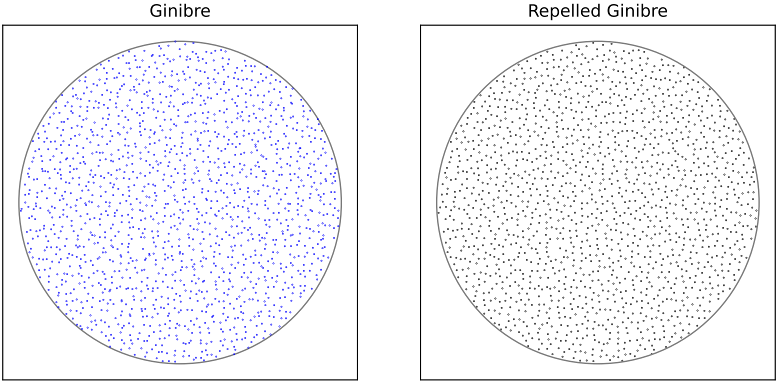
Experiments



Variance for Poisson ($d = 3$)

D. Hawat, R. Bardenet, and R. Lachièze-Rey. *Preprint*, 2023.

Experiments



D. Hawat, R. Bardenet, and R. Lachièze-Rey. *Preprint*, 2023.

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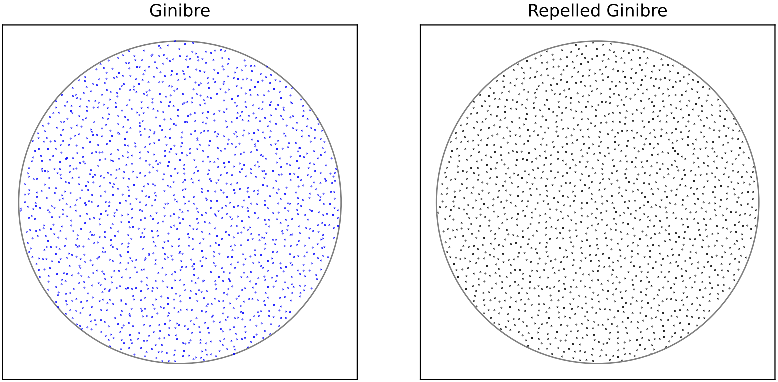
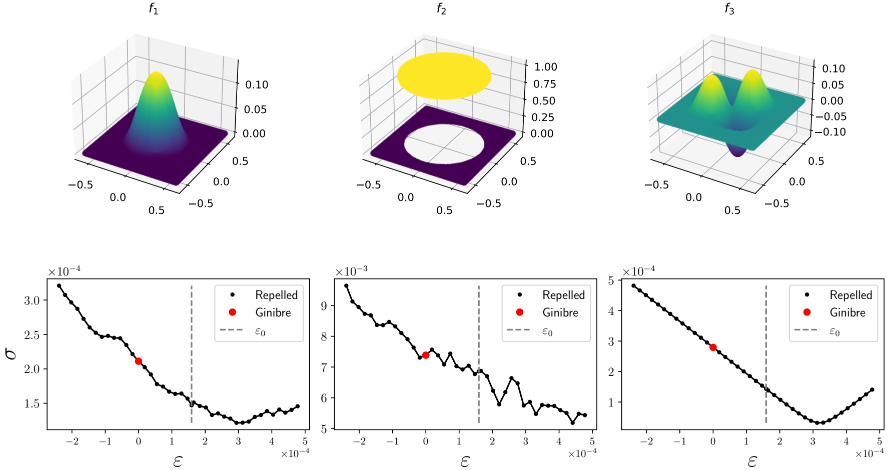


D. Hawat, G. Gautier, R. Bardenet, R. Lachièze-Rey. *Statistics and Computing*, 2023.

Experiments



Variance for Ginibre ($d = 2$)

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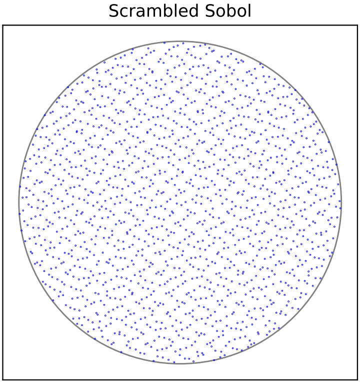
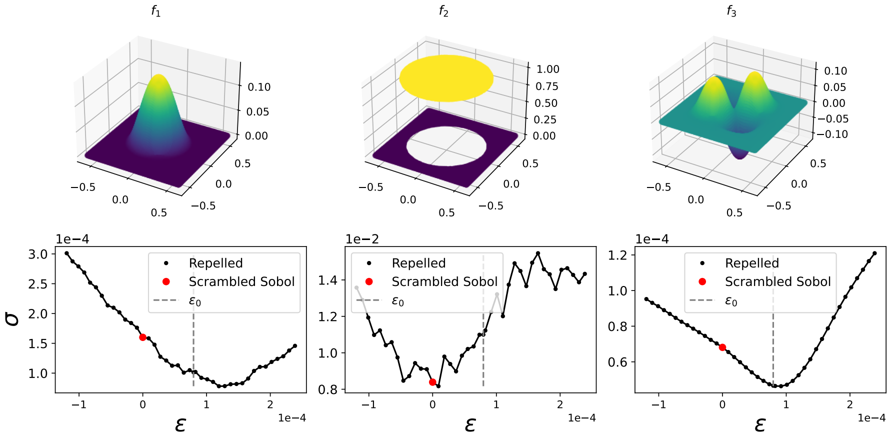
Experiments



D. Hawat, R. Bardenet, and R. Lachièze-Rey. *Preprint*, 2023.

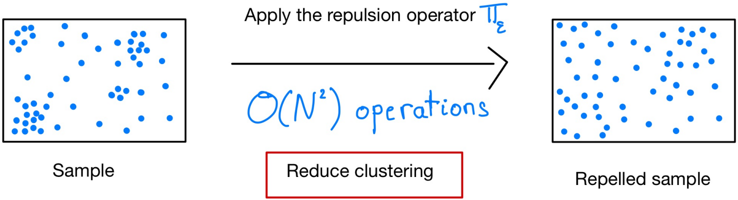
Experiments



Variance for Scrambled Sobol ($d = 3$)

D. Hawat, R. Bardenet, and R. Lachièze-Rey. *Preprint*, 2023.

Repelled point processes



$$\mathrm{Var}\left[\sum_{x \in \Pi_\varepsilon \mathcal{P}} f(x)\right] < \mathrm{Var}\left[\sum_{x \in \mathcal{P}} f(x)\right]$$

ε: positive, small

f: C^2-function of compact support

$\mathcal{P}$: homogeneous Poisson point process

Code availability

CI passing | codecov 55% | python >=3.8,<3.10

1. Open-source Python toolbox called MCRPPy.[1]
2. Available on GitHub.
3. Tutorial Jupyter notebook.[2]

MCRPPy

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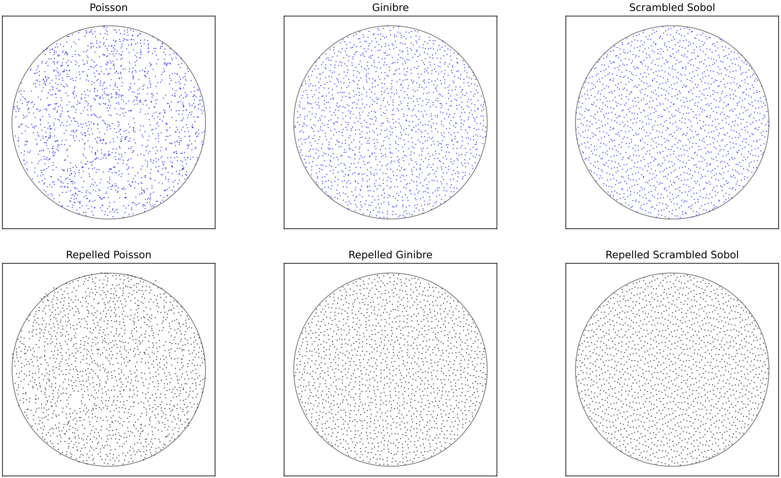
[1] https://github.com/dhawat/MCRPPy

[2] https://github.com/dhawat/MCRPPy/tree/main/notebooks

Perspectives

- Prove variance reduction for stationary point processes.
- Generalize to non-homogeneous PPPs.
- Study the attractive case of the operator ($\varepsilon < 0$).
- Derive an adequate boundary correction.

THANK YOU!



D. Hawat, R. Bardenet, and R. Lachieze-Rey. Repelled point processes with application to numerical integration, *Preprint, Axiv, HAL*, 2023.