Master 2 Thesis

Mathematics modeling and machine learning

RANDOM ALLOCATION AND GRAVITATIONAL ALLOCATION FROM LEBESGUE TO THE GINIBRE POINT PROCESS

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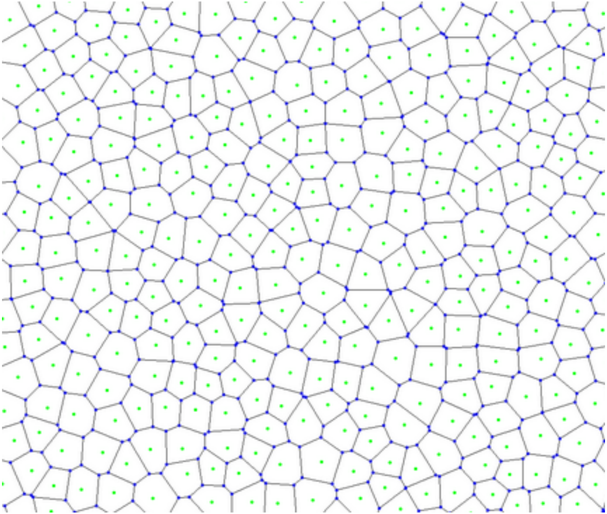
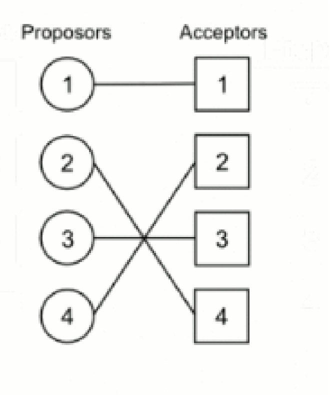
Transportation, Matching, Allocation:

Given two measures ν and μ on Λ

- A transportation between ν and μ is a measure ρ on $\Lambda \times \Lambda$ whose first marginal is ν and the second marginal is μ.
- When ν and μ are both counting measures, the transportation will be also called a matching.
- When ν is the Lebesgue measure and μ is a counting measure, the transportation will be called an allocation.

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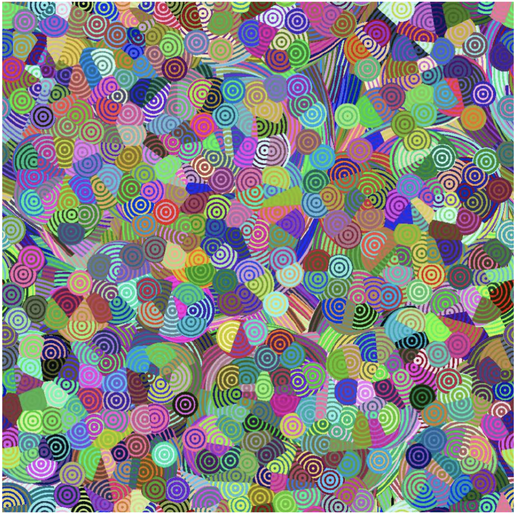

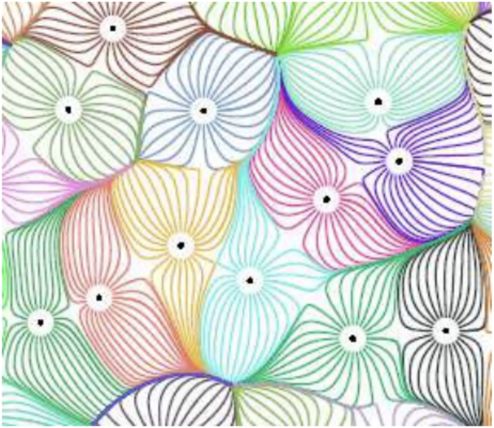

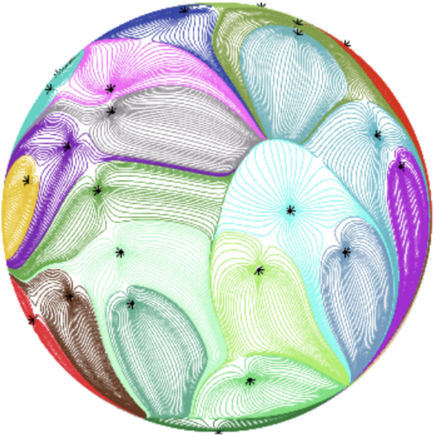

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Gravitational allocation from Lebesgue to the Poisson point process:

- Let $\mathcal{Z}$ be a standard Poisson point process on $\mathbb{R}^d$
- Consider the vector function

$$F(x) := \sum_{z \in \mathcal{Z}, |z-x| \uparrow} \frac{z-x}{|z-x|^d},$$

to be the force of attraction acting on each point x of $\mathbb{R}^d \setminus \mathcal{Z}$.

- For any $x \in \mathbb{R}^d \setminus \mathcal{Z}$, consider the integral curve $Y_x(t)$, to be unique solution of

$$\frac{dY_x(t)}{dt} = F(Y_x(t)), \;\; Y_x(0) = x.$$

define for some maximal time $\tau_x \in]0\,,\,\infty]$.
We call these curves the gravitational flow curves.

Proposition:
Assume $d \geq 3$. Almost surely, the above series defining the force function

$$F(x) := \sum_{z \in \mathcal{Z}, |z-x| \uparrow} \frac{z-x}{|z-x|^d},$$

converges simultaneously for all x for which it is defined, and defines a translation-invariant (in distribution) vector valued random function. The random function F is almost surely continuously differentiable where it is defined.

- We detailed the proof providing some steps which were hidden in the original proof. (Page 8).
- We gave another way to prove the previous propriety. (Page 13).
- We proved that the force function F, is divergent in dimension $d = 2$. (Page 12).

$\boxed{\text{Alternative formulation of } F:}$

$$F(x) := \sum_{z \in \mathcal{Z}, |z-x| \uparrow} \frac{z-x}{|z-x|^d},$$

We detailed the proof of equation (1) providing some steps which were hidden in the original proof. (Page 14).

$$\sum_{z \in \mathcal{Z}, \, |z-u| \uparrow} \frac{z-x}{|z-x|^d} - \sum_{z \in \mathcal{Z}, \, |z-v| \uparrow} \frac{z-x}{|z-x|^d} = \kappa_d (u-v). \qquad (1)$$

This equation lead to a new formulation of the force function F as follow

$$F(x) := \sum_{z \in \mathcal{Z}, |z| \uparrow} \frac{z-x}{|z-x|^d} + \kappa_d x.$$

1. Not all phenomenon could be modeled by the Poisson point process.
2. The Ginibre point process has the anti-clustering or repulsion built into its definition.
3. It has never been discussed before.
4. We claimed that the gravitational allocation to the Ginibre point process will be the most economic gravitational allocation.
5. The construction of a gravitational allocation to the Ginibre is not as simple as that to the Poisson.
6. The gravitational allocation to a finite set of point of the Ginibre is also interesting, and has never been discussed before.

Ginibre point process:

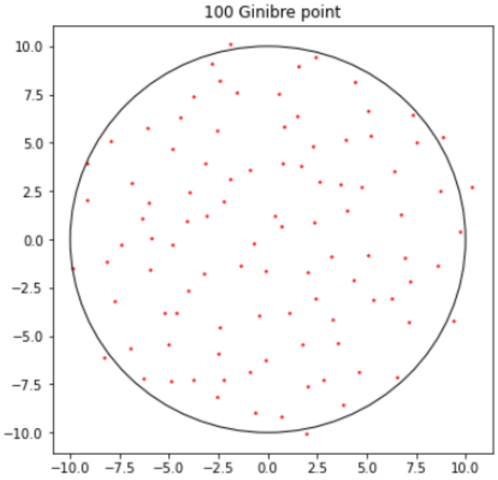
The Ginibre point process, is defined as the Determinantal point process on $\mathbb{C}$ with integral kernel

$$\mathbb{K}(z_1, z_2) = \frac{1}{\pi} e^{z_1 \bar{z}_2} e^{-\frac{1}{2}(|z_1|^2 + |z_2|^2)}, \qquad z_1\,,\ z_2\ \in \mathbb{C},$$

with respect to the Lebesgue measure on $\mathbb{C}$.

Finite Ginibre point process:

Let M be an $N \times N$ matrix with i.i.d. standard complex Gaussian entries.
Then, the eigenvalues of M form a Determinantal point process on the complex plane which converges to the Ginibre point process, and it's called the **finite Ginibre point process.**


100 Ginibre point
10.0
7.5
5.0
2.5
0.0
−2.5
−5.0
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0.0
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5.0
7.5
10.0

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Truncated Ginibre Point process:

- We are not forced to simulate the conditioning on being N points.
- we introduce a new kernel, as well as the associated point process $\mathcal{G}^N$ by setting:

$$\mathbb{K}^N_{cond}(z_1, z_2) = \sum_0^{N-1} \phi^N_n(z_1)\bar{\phi}^N_n(z_2), \qquad z_1,\ z_2 \in \mathbf{B}(0, R),$$

where, ϕ^N_n corresponds to the function ϕ_n restricted to the compact $\mathbf{B}(0, \sqrt{N})$ (after renormalization).

Gravitational allocation from Lebesgue to the Truncated Ginibre point process:

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- Let $\mathcal{G}^N = \{z_1, z_2, ..., z_N\}$ be a realisation of N points of the Truncated Ginibre ensemble in the complex plane $\mathbb{C}$.
- For all $x \in \mathbb{C}$, let

$$f(x) := \prod_{k=1}^{N} (x - z_k),$$

so that $\mathcal{G}^N = f^{-1}\{0\}$.

- The potential function:

$$u(x) = \log|f(x)| - \frac{1}{2}|x|^2.$$

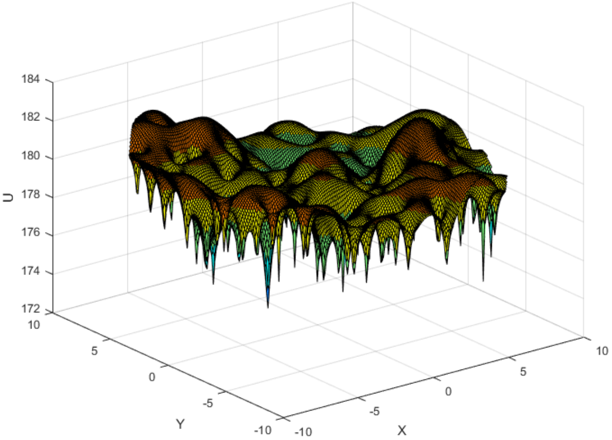
- Define the force function to be:

$$F(x) = -\nabla u(x).$$

- For any $x \in \mathbb{C} \setminus \mathcal{G}^N$ consider the integral curve $Y_x(t)$ defined as solution of the differential equation

$$\frac{dY_x}{dt}(t) = F(Y_x(t)) = -\nabla u(Y_x(t)), \; Y_x(0) = x. \qquad (2)$$

- $u(x) = \log|f(x)| - \frac{1}{2}|x|^2$.
- $\Delta u = 2\pi \sum_{x \in \mathcal{G}^N} \delta_x - 2$, in distributional sense.



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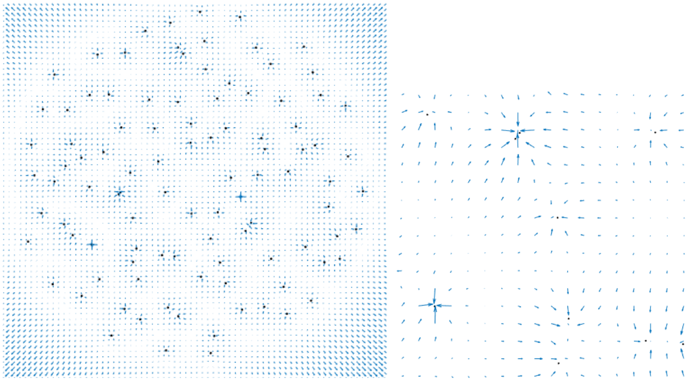
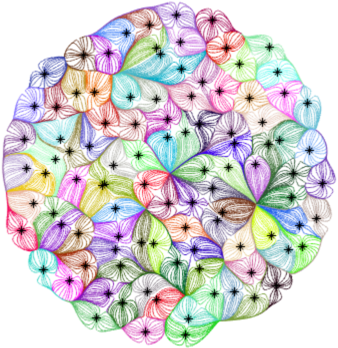
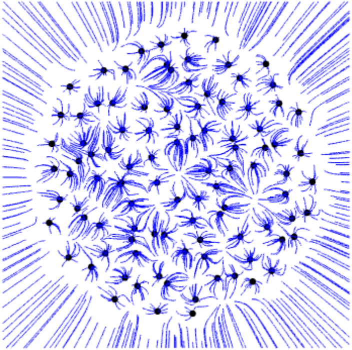
- $F(x) = -\nabla u(x)$.
- $\frac{dY_x}{dt}(t) = F(Y_x(t)) = -\nabla u(Y_x(t)), \; Y_x(0) = x$.







- If we define

$$u_c(x) = \log|f(x)| - \frac{1}{2} \times c \times |x|^2,$$

  to be the potential function, then each star will have a basin of area π/c. In our case, each basin will have a volume equal to π.
- This is a result of the divergence theorem, but apply it we have to proof that the basins are bounded by finitely many smooth curves.

Proofs:

- We proved that each basin of attraction due to the allocation to the Truncated Ginibre has volume equal to π based on Liouville's theorem. (Page 33).
- We proved that the basins due to the Gravitational allocation to the Truncated Ginibre point process are bounded and

$$\bigcup_{k=1}^{N} \mathcal{B}(z_k) \subsetneq \mathbf{B}(0, \frac{1+\sqrt{5}}{2}\sqrt{N}).$$

(Page 34).

- We gave a way to construction a gravitational allocation from the Lebesgue measure to the Ginibre point process. (Page 24).
- We gave a proof based on the divergence theorem of the conditional fairness of the allocation to the Ginibre point process. (Page 29).

Simulation:

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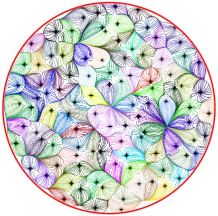
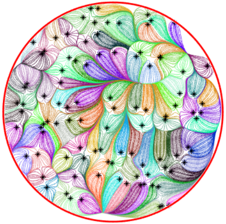
Algorithm 1 Gravitational allocation

```
Sample N point of the Truncated Ginibre 𝒢^N = {z_1, ..., z_N}.
Define f(x) := ∏_{i=1}^{N} (x − z_i).
Sort 𝒢^N in increasing order, let {Z_1, ..., Z_N} be 𝒢^N reordered.
Let [X, Y] be the grid containing the compact B(0, √N).
let u(x) := log|f(x)| − 0.5|x|^2 be the potential function.
Evaluate u at X + iY.
[Dx, Dy] ← gradient(u)
Set s = (sx, sy) a set of equispaced points of distance ϵ.
for i = 1 → N do
    h ← streamline(X, Y, Dx, Dy, real(Z_k)+sx, imag(Z_K)+sy)
    Plot Z_k
    Plot h
end for
```

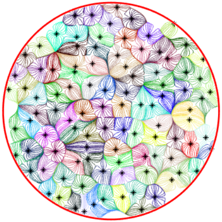
PERSPECTIVE



The allocation of the disk to 100 Ginibre points.



The allocation of the disk to 100 Uniform points.



The allocation of the disk to 100 zeros of the GAF.

THANK YOU.