MAP 5 Laboratory

University of Paris

Master 2 thesis

Mathematics, Modeling, and Machine Learning track

Random allocations and Gravitational allocation to the Ginibre Ensemble

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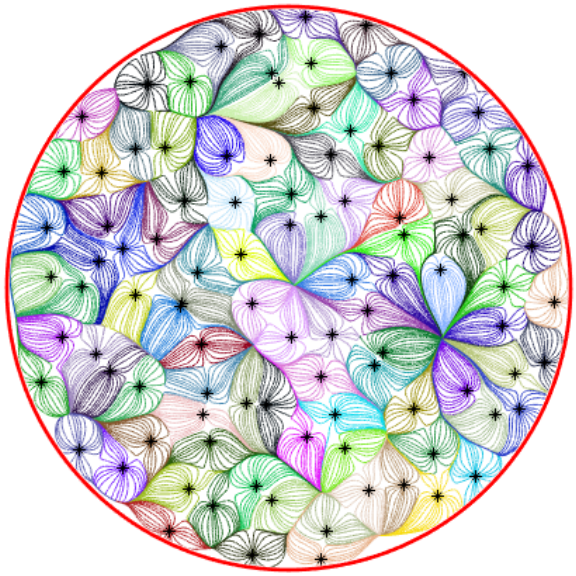
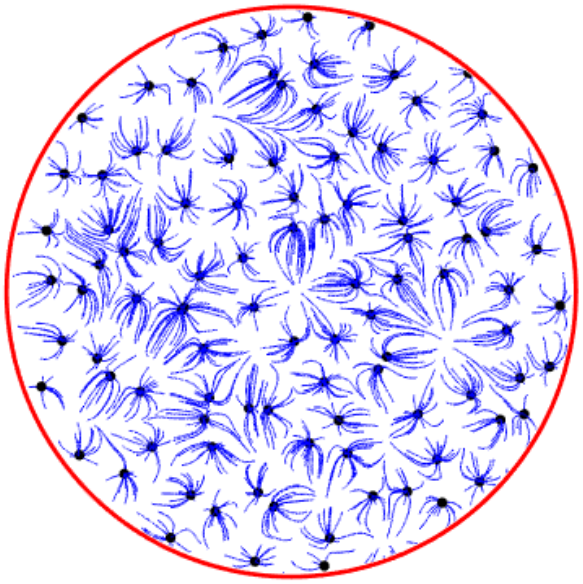
Abstract

In this Master 2 Thesis, we talk about gravitational allocations, we prove that the force defining the gravitational allocation of Lebesgue to the Poisson process is divergent in dimension 2, convergent in dimension ≥ 3, and we detail some proofs.
Furthermore, we construct a gravitational allocation of Lebesgue to the complex Ginibre Ensemble. As in practice, we have first to construct a finite model and then let the model goes to infinity. Hence, it's very important to study the finite version of our allocation, so we restrict our construction to the Truncated Ginibre point process projected on the Disk. We prove the fairness of this transportation, as well as that the basins are bounded, and that the union of the basins associated to N points of Truncated Ginibre point process, is contained in the ball centered at the origin and of radius $\frac{1+\sqrt{5}}{2}\sqrt{N}$.
We simulate the allocation and we give a proposition of an algorithm to simulate and visualize this allocation.





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Laboratory presentation

MAP5 is attached to the University of Paris (formerly Paris Descartes University) and to the National Institute of Mathematical Sciences and their Interactions (INSMI) of the CNRS. It is a part of the Mathematical Sciences Foundation of Paris (FSMP).
The MAP5 Laboratory, contains 4 researches fields, probabilities, statistics, image processing and digital modeling.

1 Introduction:

The stable marriage problem (also a stable matching problem or SMP) is the problem of finding a stable matching, between two equally sized sets of elements given an ordering of preferences for each element. In 1962, David Gale and Lloyd Shapley proved that for any, equal number of men and women, it is always possible to solve the Stable Marriage Problem, and make all marriages stable; also, they presented an algorithm to do so. Algorithms for finding solutions to the stable marriage problem have applications in a variety of real-world situations, perhaps the best known of these being in the assignment of graduating medical students to their first hospital appointments.
In 2006, Hoffman, Holroyd, and Peres have introduced the concept of **stable allocation**, in their article "A stable marriage of Poisson and Lebesgue" [3]. They have talked about a method of *stable matching*, in the sense of Gale-Shapley stable marriage problem, and until now this is the only stable allocation constructed.
The existence of well-localized transportation of the area measure $\frac{1}{\pi}\mathcal{L}^2$ (where $\mathcal{L}^2$ represent the Lebesgue measure in the plane), to the set $\mathcal{Z}_f$, (which represent the random set of zeros of the Gaussian Entire Function), was first studied by Sodin and Tsirelson in [10]. Using the Hall matching lemma, and some potential theory, the authors of [10] proved the existence of transportation with subgaussian decay of the tail probability. This work has been followed by Sodin, Nazarov, and Volberg in [9]. The goal of the authors was to carry out another approach, which was also suggested but not followed in [10], namely, transportation via the gradient flow. They proved that the probability that a given point z, lies at a distance larger than R, from the zero that it is attracted to, decay as $e^{-R^{8/5}}$. This transportation was the first work on gravitational allocation's field without naming it as a gravitational allocation.
In 2018 a new work in the field of allocation has been published: "The gravitational allocation to Poisson process " [2]. In this work, the allocation rule was based on the Gravitational flow field.
The construction of this allocation was different from the allocation described in 2006 [3], and similar to the transportation built by Sodin, Nazarov, and Volberg [9]. This time, we didn't force the basins to allocate a cell of volume 1 but, as the construction follows the Newtonian law of gravity then, when the potential takes a particular form which mimics the gravitational potential of Newtonian mechanics, it is ensured that a.s. each cell will have an area equal to 1. Indeed, they proved that the force defining the gravitational allocation from Lebesgue to the Poisson point process in $\mathbb{R}^d$ is convergent, for $d \geq 3$. They prove also that this allocation rule is economic, in the sense that the allocation diameter defined as the diameter X of the basin of attraction containing the origin, is a random variable with a rapidly decaying tail. Specifically, the tail bound $\mathbb{P}(X > R) \leq C \exp[-cR(\log R)^{\alpha_d}]$, for all $R > 2$, where $\alpha_d = \frac{d-2}{d}$ for $d \geq 4$; α_3 can be taken as any number $< -4/3$, and C, c are positive constants that depend on d and α_d.
Also, Holroyd and Peres, in their paper [17], have shown that: if $d = 1$ or 2 then, for every allocation rule from Lebesgue to a Poisson point process (PPP) with unit intensity, the allocation diameter X satisfies $\mathbb{E}X^{d/2} = \infty$. In this case, the decay of $\mathbb{P}(X > R)$ to 0 cannot be faster than polynomial in R so that in dimension 1 and 2 the allocation diameter may goes to infinity. Nevertheless, one can study the behavior of gravitational allocation in two dimensions by restricting the work onto a finite set of points.
So based on the idea of considering a finite set of point in $\mathbb{R}^2$, in 2019 another paper on gravitational allocation has been published: "Gravitational allocation for uniform points on the sphere" [6], which has the same concept as the gravitational allocation form Lebesgue to the Poisson process. The authors proved that the expected distance between a point on the sphere and the associated point of $\mathcal{L}$, (where $\mathcal{L}$ represent a set of n independent uniform point on the sphere S_2), is $\mathcal{O}(\sqrt{\log(n)})$. Moreover, they used this result to define an optimal matching, between two collections of n independent and uniform points on the sphere, and they proved that the expected distance between a pair of matched points is $\mathcal{O}(\sqrt{\log(n)})$, which is optimal.

2 Plan of work

The article "Gravitational allocation to Poisson points" give sketches for some of the proofs.
In this work, we write explicitly the full proofs of the existence of the force field (subsection 4.2) and the rearrangement identity (subsection 4.3) for the gravitational allocation from Lebesgue to the Poisson point process also, we give another proof of the convergence of the force field in dimension ≥ 3, and we prove the divergence in dimension 2 (subsection 4.2).
In section 5, we give some definition, and properties that we will need to define the Laplace operator of the potential function in section 7.
In section 6, we try to understand the strategies to follow, and the conditions to provide, to build a fair allocation.
In section 7, we construct a gravitational allocation to the Ginibre point process (subsection 7.3), which has never been discussed before and we give a proof 7.3.1 that our construction is a fair allocation conditional on having each basins bounded by finitely many smooth curves.
As in practice, an infinite construction is useless without having a finite model, so we look forward constructing a gravitational allocation from the Lebesgue to a finite set of Ginibre point process but, working with a finite set of points of the Ginibre point process is not as simple as a finite set of points of the Poisson point process, since the number of point of the Ginibre that falls in a compact is arbitrary, and we can't hope to control them, if we didn't modify the Kernel. So we consider the Truncated Ginibre $\mathcal{G}^N$ restricted on a compact space (subsection 7.4), and give an algorithm to simulate it described in [14]. Then, we construct a transportation by gradient flow, from Lebesgue to the Truncated Ginibre point process (subsection 7.5). Moreover, we give explicit proof that it's a fair allocation (7.5), and we prove that the basins due to this allocation are bounded, and that the union of the basins associated to N points of Truncated Ginibre point process, is strictly contained in the ball centered at the origin and of radius $\frac{1+\sqrt{5}}{2}\sqrt{N}$, (7.5). Also we mention how we can control the volume of the basins. Furthermore, we propose an algorithm to simulate and visualize our construction (subsection 7.6), and we visualize the potential function, the basins, the force field, and the trajectory of a set of points under our flow field. Finally, we project this construction on the Disk $\mathbf{D}(0, \sqrt{N})$ (subsection 7.7), and we open new perspectives.

3 Preliminary

3.1 Point Process

Definition 3.1.1. "Random closed set and point process" [1]
A random closed set is a measurable mapping from a probability space $(\Omega, \mathcal{A}, \mathbb{P})$ into $(\mathbb{F}, \Sigma')$, where:

1. $\mathbb{F}$ is the set of all closed sets of $\mathbb{R}^d$.
2. Σ' is the $\sigma-$algebra generated over $\mathbb{F}$ by

$$\mathbb{F}_K = \{F \in \mathbb{F}; F \cap K = \emptyset\},$$

for all compact $K \subset \mathbb{R}^d$.

Let $\mathbb{M} = \{F \in \mathbb{F} \text{ s.t. } \forall \text{ compact } K \subset \mathbb{R}^d, Card(F \cap K) < \infty\}$.
A **point process** is a random closed set $\mathcal{M}$ such that:

$$\mathbb{P}(\mathcal{M} \in \mathbb{M}) = 1.$$

If a point process $\mathcal{M}$ assigns at most measure 1 to singletons then it is called a **simple point process.**

The Poisson process [1] (denoted PPP; we also denote a Poisson point process of intensity λ by $PPP(\lambda)$), was considered as the most important point process to study physical phenomena being characterized by its independence when restricted to disjoint subsets of the underlying space. But not all phenomena can be modeled by the Poisson point process.
For instance if one looks at outbreaks of a rare disease in a province, then knowing that there is a case in a particular location makes it more likely that there are more such cases in a neighborhood of that location. On the other hand, if one looks at the distribution of like-charged particles confined by an external field (physicists call it a "one component plasma" [22]), then the opposite is true. Knowing that a particular location holds a particle makes it unlikely for there to be any others close to it. These two examples indicate two ways of breaking the independence assumption, inducing more clumping ("positively correlated" 3.1.2) as in the first example or less clumping ("negatively correlated") as in the second.
Many natural questions arise. "Are there interesting point processes that have less clumping than Poisson processes?"
One natural way of getting such a process without putting in the anti-clumping property by hand is to extract zero sets of random polynomials or analytic functions, for instance, zeros of random polynomials with stochastically independent coefficients.
Furthermore, Determinantal processes (denoted DPP) have this anti-clustering or repulsion built into their definition.

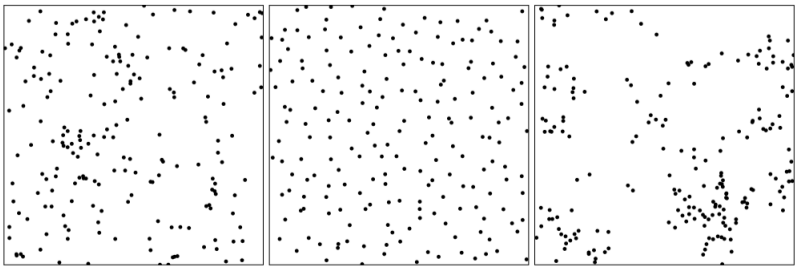


Figure 1: Samples of translation invariant point processes in the plane: Poisson (left), Determinantal (center) and Permanental (right). Determinantal processes exhibit repulsion, while Permanental processes exhibit clumping. All the three samples shown are portions of certain translation invariant point processes in the plane, with the same average number of points per unit area (from [4]).

Definition 3.1.2. "Correlation functions or joint intensities"
Let $\mathcal{M}$ be a simple point process on Ω.
The joint intensities (or correlation functions) of $\mathcal{M}$ with respect to a measure μ, are functions (if any exists) $\rho_k : \Omega^k \longrightarrow [0, \infty[$ for $k \geq 1$, such that for any family of mutually disjoint subsets $D_1, ..., D_k$ of Ω,

$$\mathbb{E}\Big[\prod_{i=1}^{k} \mathcal{M}(D_i)\Big] = \int_{\prod_i D_i} \rho_k(x_1, ..., x_k) d\mu(x_1)...d\mu(x_k).$$

In addition, we shall require that $\rho_k(x_1, ..., x_k)$ vanish if $x_i = x_j$ for some $i \neq j$.
N.B. The distribution of a point process can be described by its joint intensities.

3.2 Transportation, matching and allocation

Given two measures ν and μ on Λ, a transportation between ν and μ is a measure ρ on $\Lambda \times \Lambda$ whose first marginal is ν and the second marginal, μ. When ν and μ are both counting measures (i.e., atomic measures with atoms of size 1), and so is ρ, the transportation will be also called a matching. When ν is the Lebesgue measure and μ is the counting measure (or when ν is a fixed

deterministic measure and μ is a point process), the transportation will be called an allocation. Informally we think of ρ as taking a mass $d\nu(x)$ from the point x and spreading it over μ by transporting a mass of $\rho(x, dy)$ to the point y. A matching is just what it says, a pairing of the support of ν with the support of μ (when both are counting measures). An allocation may be picturesquely described as a scheme for dividing up land (Lebesgue measure) among farmers (points of the point process) in a fair manner. In other words, for $\mathcal{Z}$ a random point process, the whole space $\mathbb{R}^d$ is split into disjoint random sets (or basins) $\mathcal{B}(a)$ of equal Lebesgue measure indexed by $a \in \mathcal{Z}$.

3.3 Some definitions and notations

Let $d \in \mathbb{N}$ and $\Xi \in \mathbb{M}$.
The elements of Ξ will be referred to as **centers** (or stars), and the other points of $\mathbb{R}^d$ will be called **sites**.
An **allocation** (of Lebesgue measure to Ξ) is a measurable function $\psi : \mathbb{R}^d \to \Xi \cup \{\infty\}$, that satisfies:

1. $\mathrm{Vol}(\psi^{-1}(\infty)) = 0$
2. $\mathrm{Vol}(\psi^{-1}(z)) = c, \quad \forall z \in \Xi,$

where c is a constant, and $\mathrm{Vol}(\cdot)$ is the Lebesgue measure on $\mathbb{R}^d$ for $z \in \Xi$. We call $\psi^{-1}(z)$, **the cell allocated** to z or the **territory** of the center z.
Let $\mathcal{Z}$ be a translation-invariant simple point process in $\mathbb{R}^d$ with unit intensity defined on some probability space $(\Omega, F, \mathbb{P})$ i.e. for any open set $A \subseteq \mathbb{R}^d$, the random variable $\mathrm{Card}(A \cap \mathcal{Z})$ has mean $\mathrm{Vol}(A)$, and for any $x \in \mathbb{R}^d$ and that for any open sets $A_1, A_2, \ldots, A_k \subseteq \mathbb{R}^d$, the random vector

$$(\mathrm{Card}((A_1 + x) \cap \mathcal{Z}), \mathrm{Card}((A_2 + x) \cap \mathcal{Z}), \ldots, \mathrm{Card}((A_k + x) \cap \mathcal{Z}))$$

has a distribution that does not depend on x.
An allocation rule (from Lebesgue measure to $\mathcal{Z}$) is a mapping $\mathcal{Z} \mapsto \psi_{\mathcal{Z}}$ that is defined $\mathbb{P}$-a.s., measurable (with respect to the relevant σ-algebra), and such that:

1. almost surely $\psi_{\mathcal{Z}}$ is an allocation of the Lebesgue measure to $\mathcal{Z}$
2. the mapping $\mathcal{Z} \mapsto \psi_{\mathcal{Z}}$ is translation-equivariant, in the sense that $\mathbb{P}$-a.s. for any $x, y \in \mathbb{R}^d$ we have $\psi_{\mathcal{Z}+x}(y + x) = \psi_{\mathcal{Z}}(y) + x$.

An allocation rule may satisfy numerous additional properties, for example the cells $\psi_{\mathcal{Z}}^{-1}(z)$ may be open, may contain their center, they may be bounded, each cell may be connected.
If all the cells are bounded, we introduce the concept of **allocation diameter**, which is the random variable

$$X = \mathrm{diam}(\psi_{\mathcal{Z}}^{-1}(\psi_{\mathcal{Z}}(0)))$$

where $\mathrm{diam}(\cdot)$ denotes the diameter of a set [1] .
To measure how economical our allocation rule is, we look at the rate of decay of the tail $\mathbb{P}(X > R)$ of the distribution of X as $R \to \infty$, intuitively, a fast rate of decay means that it is rare for the points to be allocated to a far location.

[1] For a subset A of $\mathbb{R}^d$ $\mathrm{diam}(A) = \sup\{d(x, y) : x, y \in A\}$, where d is the euclidean metric.

4 Gravitational allocation from Lebesgue to the Poisson points process

4.1 Construction

The construction was inspired from the idea of the transportation by gradient flow of Lebesgue to the zero of the Gaussian entire function constructed by Nazarov, Sodin and Volberg en 2007.
Let $\mathcal{Z}$ be the standard Poisson point process in $\mathbb{R}^d$, with $d \geq 3$. For concreteness we denote $\mathcal{Z} = (z_i)_{i=1}^{\infty}$ for specific ordering of point of $\mathcal{Z}$ by increasing distance from 0. We attribute to each center $z \in \mathcal{Z}$ a force of attraction.
Consider the random vector $F : \mathbb{R}^d \to \mathbb{R}^d$ defined by:

$$F(x) := \sum_{z \in \mathcal{Z},\, |z-x| \uparrow} \frac{z-x}{|z-x|^d}, \tag{4.1.1}$$

to be the force of attraction acting on each point x of $\mathbb{R}^d \setminus \mathcal{Z}$, and let the allocation rule be defined by the function F that maps each site x to the sum of the forces of attraction of the centers, on x, where the summands are arranged in order of increasing distance from x.
The term $(z-x)/|z-x|^d$ represents a gravitational force felt by a unit mass at a point x due to the influence of a unit mass placed at a point z. When $d = 3$, this is the ordinary Newtonian gravitational force.
When the potential takes a particular form that mimics the gravitational potential of Newtonian mechanics, it is ensured that each cell has area 1. Moreover, here we consider the force as a vector field (as a velocity rather than an acceleration), thus we sum on an increasing distance from the target point x following the concept of having an "increasing field ".
For any $x \in \mathbb{R}^d \setminus \mathcal{Z}$, consider the integral curve $Y_x(t)$, to be unique solution of

$$\frac{dY_x(t)}{dt} = F(Y_x(t)),\ Y_x(0) = x.$$

define for some maximal time $\tau_x \in]0\,,\,\infty]$.
We call these curves the gravitational flow curves (in a simple inertialess Newtonian gravitational world).
For each center $z \in \mathcal{Z}$, we say that the curve Y_x ends at z, if $\lim_{t \to \tau_x} Y_x(t) = z$, and define the basin of attraction of z by

$$\mathcal{B}(z) = \{x \in \mathbb{R}^d \setminus \mathcal{Z},\, Y_x(t) \text{ ends at } z\} \cup \{z\}.$$

The **gravitational allocation rule** will be defined as follow :

$$\psi_z(x) = \begin{cases} z & x \in \mathcal{B}(z) \text{ for } z \in \mathcal{Z}, \\ \infty & x \notin \cup_{z \in \mathcal{Z}} \mathcal{B}(z). \end{cases}$$

4.2 Simultaneous convergence and differentiability :

Proposition 4.2.1. Assume $d \geq 3$. Almost surely, the series (4.1.1) converges simultaneously for all x for which it is defined, and defines a translation-invariant (in distribution) vector valued random function. The random function F is almost surely continuously differentiable where it is defined.

S. Chandrasekhar observes that this series never converges absolutely but, conditionally it converges in dimension 3 and higher, and that the order of the summation is important to get the

convergence, following the concept of having an "increasing field", (as we already have mentioned, here we consider the force as a vector field) thus, the idea of summing over an increasing distance from the target point x.
(In the original article [2] we were given a sketch of the proof so here we write the proof explicitly of the existence, for the proof of the differentiability we refer to [2]).
The proof is based on the Kolmogorov's three-series theorem, which gives a criterion for the almost sure convergence of an infinite series of random variables in terms of the convergence of three different series involving properties of their probability distributions. We'll state the theorem before going through the proof.

Theorem 4.2.1. "The Kolmogorov's Three-Series theorem"
Let $(X_n)_{n\in\mathbb{N}}$ be independent random variables. The random series $\sum_{n=1}^{\infty} X_n$ converges almost surely in $\mathbb{R}$ if the following conditions hold for some $A > 0$, and only if the following conditions hold for any $A > 0$:

1. $\sum_{n=1}^{\infty} \mathbb{P}(|X_n| \geq A)$ converges.
2. Let $Y_n = X_n 1_{\{|X_n|\leq A\}}$, then $\sum_{n=1}^{\infty} \mathbb{E}[Y_n]$, the series of expected values of Y_n, converges.
3. $\sum_{n=1}^{\infty} V(Y_n)$ converges.

Proof. of proposition 4.2.1
As the sum is defined in a translation-invariant manner so, by showing that $F(0)$ converges we get that a.e. a.s. the sum defining F converges; by continuity we conclude that a.s. a.e. the sum defining F is convergent.
Indeed, $\mathcal{Z}$ is a translation-invariant simple point process in $\mathbb{R}^d$ so, for any open sets $A_1, ..., A_k \subseteq \mathbb{R}^d$ the random vector $(|(A_1 + x) \cap \mathcal{Z}|, ..., |(A_k + x) \cap \mathcal{Z}|)$ has a distribution that doesn't depends on x thus, it's enough to show the convergence of F at $x = 0$,

$$F(0) = \sum_{\substack{z\in\mathcal{Z}\\ |z|\uparrow}} \frac{z}{|z|^d} \tag{4.2.2}$$

The idea is to use the Kolmogorov three series theorem to show the convergence, so we need to define a truncated series having a finite sum of expectation, and variance.
For concreteness, we denote $\mathcal{Z} = (z_i)_{i=1}^{\infty}$ for the specific ordering of the points of $\mathcal{Z}$ by increasing distance from 0, and we define $(\rho_i)_{i\geq 0}$ to be the associated sequence of distances such that, $\rho_0 = 0$ and $\rho_i = |z_i|$ for $i \geq 0$.
As, $\mathcal{Z}$ is a standard Poisson point process then, the random variable $(\kappa_d(\rho_i^d - \rho_{i-1}^d))_i$ are i.i.d. and follow the exponential distribution of parameter 1 with, $\kappa_d = Vol(B(0,1)) = \pi^{d/2}/\Gamma(d/2+1)$, where Γ is the Gamma function.

Claim 4.2.1. If $\mathcal{Z} = (z_i)_{i=0}^{\infty}$ is a $PPP(1)$ so, $(|z_i|^d)_i \rightsquigarrow PPP(\kappa_d)$.

Proof of claim 4.2.1
Let

$$\begin{aligned} \phi\colon \mathbb{R}^d &\to \mathbb{R}_+ \\ z &\mapsto |z|^d \end{aligned}$$

Let K be compact of $\mathbb{R}_+$ and , $K' = \{z_i \in \mathcal{Z} \text{ s.t. } |z_i|^d \in K\} = M^{-1}(K)$, where $M = \phi \circ \mathcal{Z}$.
First, compute $\mathcal{L}^d(K')$, where $\mathcal{L}^d$ denotes the Lebesgue measure over $\mathbb{R}^d$:
We'll use the theorem mentioned by Stein in the appendix from his book " Fourier analysis" which

says :
"Let: $f(|x|)$ be integrable radial function in $\mathbb{R}^d$, then:

$$\int_{\mathbb{R}^d} f(|x|)dx = w_{d-1}\int_0^\infty f(r)r^{d-1}dr$$

where, w_{d-1} is the surface of the $d-1$ unit sphere".
Let:

$$\begin{aligned} f\colon \mathbb{R} &\to \mathbb{R} \\ r &\mapsto 1_{\{r^d\in K\}} \end{aligned}$$

we have :

$$\begin{aligned} \mathcal{L}^d(K') &= \int_{\mathbb{R}^d} 1_{\{z_i\in K'\}}dz_i = \int_{\mathbb{R}^d} 1_{\{|z|_i^d\in K\}}dz_i \\ &= \int_{\mathbb{R}^d} f(|z_i|)dz_i \\ &= w_{d-1}\int_0^\infty f(r)r^{d-1}dr \\ &= \frac{w_{d-1}}{d}\int_0^\infty 1_{\{v\in K\}}dv \\ &= \frac{d\kappa_d}{d}\int_0^\infty 1_{\{v\in K\}}dv \\ &= \kappa_d\mathcal{L}(K) \end{aligned}$$

Note that $S^{d-1} = d\times V^d$, where S^d represente the surface of the d-sphere and V^d the volume of the d-ball.
As ϕ is continuous (so measurable) and, $\mathcal{Z}\rightsquigarrow PPP$ so, M is a random closed set thus, its law is defined by its capacity function T_M [1].
Now, let's find the law of M.

$$\begin{aligned} T_M(K) &= \mathbb{P}(M\cap K\neq\emptyset) \\ &= \mathbb{P}(\phi\circ\mathcal{Z}\cap K\neq\emptyset) \\ &= \mathbb{P}(\mathcal{Z}\cap\phi^{-1}(K)\neq\emptyset) \\ &= 1-\mathbb{P}(\mathcal{Z}\cap\phi^{-1}(K)=\emptyset) \\ &= 1-\mathbb{P}(Card(\mathcal{Z}\cap\phi^{-1}(K))=0) \\ &= 1-\exp(-\mathcal{L}^d(K')) \\ &= 1-\exp(-\kappa_d\mathcal{L}^d(K)). \end{aligned}$$

so, M $\rightsquigarrow PPP(\kappa_d)$. ◇
Therefore, we have $(|z_i|^d)_i \rightsquigarrow PPP(\kappa_d)$.

Claim 4.2.2. $(\rho_i^d-\rho_{i-1}^d)_i\rightsquigarrow Exp(\kappa_d)$ thus, $(\kappa_d(\rho_i^d-\rho_{i-1}^d))_i\rightsquigarrow Exp(1)$.

Proof of claim 4.2.2
Need : *"$X\rightsquigarrow Exp(\lambda)\Longrightarrow\lambda X\rightsquigarrow Exp(1)$".*
Let X be a random variable such that $X\rightsquigarrow Exp(\lambda)$
Or, $\mathbb{P}(\lambda X\leq x) = \mathbb{P}(X\leq\frac{x}{\lambda}) = 1-\exp(\frac{\lambda x}{\lambda}) = 1-\exp(x)$.

So, $\lambda X \rightsquigarrow Exp(1)$. $\diamond$
By the law of large number we have :

$$\frac{\rho_i}{i^{\frac{1}{d}}} = \left[\frac{\sum_{j=1}^{i} \rho_j^d - \rho_{j-1}^d}{i}\right]^{1/d} \xrightarrow[i\to\infty]{} \kappa_d^{-1/d} \tag{LLN}$$

Now, if we condition on the values of $(\rho_i)_i$ thinking of them as deterministic sequence s.t. (LLN) holds then, each z_i is distributed uniformly on the sphere of raduis ρ_i around the origin.
For any $i \geq$ each term $\frac{z_i}{|z_i|}$ in the sum in (4.2.2) has conditional mean 0 and variance bounded by $\mathcal{O}(\rho_i^{2(d-1)})$.
Indeed, let $u := \frac{z_i}{|z_i|}$, we have $u \rightsquigarrow \mathcal{U}_{S_{d-1}}$, and
$\mathbb{E}[u] = \int_{S^{d-1}} u d\mu = \int_{S_+^{d-1}} u d\mu + \int_{S_-^{d-1}} u d\mu = \int_{S_+^{d-1}} u d\mu - \int_{S_+^{d-1}} u d\mu = 0$.
Also we have,

$$\begin{aligned} V(\frac{z_i}{|z_i|^d}/(\rho_i)_i) &= \mathbb{E}\Big[\frac{z_i}{|z_i|^d}(\frac{z_i}{|z_i|^d})^T/(\rho_i)_i\Big] \\ &= \mathbb{E}\Big[\frac{z_i z_i^T}{|z_i|^{2d}}/(\rho_i)_i\Big] \\ &= \mathbb{E}\Big[\frac{1}{|z_i|^{2d-2}}/(\rho_i)_i)\Big] \\ &= \frac{1}{\rho_i^{2(d-1)}} \\ &\leq \mathcal{O}(i^{-2(d-1)/d}) \end{aligned}$$

The last equality holds since $\frac{\rho_i}{i^{1/d}} \longrightarrow \kappa_d$ which implies that $\mathcal{O}(\rho_i) = \mathcal{O}(i^{1/d})$.
But,

$$\sum_i V(\frac{z}{|z|^d}/(\rho_i)_i) \leq \sum_i \mathcal{O}(i^{-\frac{2(d-1)}{d}}) \,.$$

The last sum converges $\iff \frac{2(d-1)}{d} > 1 \iff 2d-2 > d \iff d > 2$.
So, condition on $(\rho_i)_i$ and (LLN), if $d > 2$ then the sum of the variances converges. Now, Condition on $(\rho_i)_i$, and on (LLN), the conditions of the Kolmogorov three-series theorem hold for $A = \rho_1^{1-d}$, and $x_i = \frac{z_i}{|z_i|^d}$,

1. $\sum_i \mathbb{P}(|x_i| < A) = \sum_i \mathbb{P}(|z_i|^{1-d} > \rho_1^{1-d}) = \sum_i 0 = 0$
   Indeed, as the ρ_i are increasing so, $\frac{1}{|z_i|} \leq \frac{1}{\rho_1}$ a.s.

2. $\sum_i \mathbb{E}\big[x_i 1_{\{\|x_i\| \leq A\}}\big] = \sum_i \mathbb{E}[x_i] = 0$.
   Since, $\{x_i\} = \{x_i 1_{\{\|x_i\| \leq A\}}\}$ a.s.

3. $\sum_i V\big[x_i 1_{\{\|x_i\| \leq A\}}\big] = \sum_i V[x_i]$ is convergent (already proved).

So, the Kolmogorov's three-series theorem holds thus, $F(0)$ is well defined a.s. conditioning on $(\rho_i)_i$, and on (LLN) .
As (LLN) holds a.e. a.s. the sum defining F converges, by continuity we conclude that a.s. a.e. the sum defining F is convergent. $\square$

In what follows we give a way to prove the divergence of the force function F in $\mathbb{R}^2$, and another way to prove the convergence in distribution, in $\mathbb{R}^d$ for $d \geq 3$. This time the proofs are based on Levy's theorem.
We start first by annunciating Levy's theorem and some definition needed throughout the proof.

Theorem 4.2.2. "Levy's theorem"
For a sequence of random variable $(X_n)_n$ we have:
"X_n converges in distribution to some random variable X, if and only if the sequence of characteristic function $\varphi_{X_n}(t) := \mathbb{E}(e^{itX_n})$ converges pointwise to some function $\varphi_X(t)$ continuous at the origin."
Moreover, φ_X is the characteristic function of X.

Definition 4.2.1. "Bessel function"
Bessel functions are canonical solutions $y(x)$ of Bessel's differential equation

$$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + (x^2 - \alpha)y = 0,$$

for an arbitrary complex number α, the *order* of Bessel function.
Bessel functions of the first kind, denoted as $\mathbf{J}_\alpha(x)$, are solutions of Bessel's diffrential equation that are finite at the origin for integer or positive α, and diverge as x approaches to zero for negative non-integer α. It's possible to define the function by its series expansion around $x = 0$, which can be found by applying the Frobenius method to Bessel's equation [19]

$$\mathbf{J}_\alpha(x) = \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\Gamma(m + \alpha + 1)}(\frac{x}{2})^{2m+\alpha},$$

where, Γ is the gamma function, i.e. a shifted generalization of the factorial function to non-integer values. Moreover, for integer value of α, another definition of the Bessel function is possible using integral representation as follows

$$\mathbf{J}_n(x) = \frac{1}{\pi}\int_0^\pi \cos(n\tau - x\sin\tau)d\tau = \frac{1}{2\pi}\int_{-\pi}^{\pi} \exp(i(x\sin\tau - n\tau))d\tau.$$

Furthermore, for large x we have an approximation of the Bessel function: [18]

$$\mathbf{J}_n(x) \overset{\infty}{\approx} \sqrt{\frac{2}{\pi x}}\cos(x - \frac{n\pi}{2} - \frac{\pi}{4}).$$

Proposition 4.2.2. Almost surely, the series (4.1.1) converges simultaneously for all x for which it is defined, for $d \geq 3$, and diverges for $d = 2$.

Proof. We will use Levy's theorem and the fact that $\mathcal{Z}$ is a PPP which gives a particular simplified formula for the characteristic function of F [1], which will simplify the calculus.
Let $\mathcal{Z}$ be the set of the stars.
If F exists, than F is the limit of the sequence

$$F_R(x) = \sum_{\substack{|z|\uparrow \\ z\in\mathbf{B}(0,R)\cap\mathcal{Z}}} \frac{z - x}{|z - x|^d}$$

as R goes to infinity. Or, the sum is defined in a translation invariant manner so, it's enough to show that

$$F_R(0) = \sum_{\substack{|z|\uparrow \\ z\in\mathbf{B}(0,R)\cap\mathcal{Z}}} \frac{z}{|z|^d}$$

converges for $d \geq 3$, (resp. divergent for $d = 2$).
As it's a vector so we have to show that each component F_R^k for $k = \{1, ..., d\}$ converges, (resp. at

least one component diverges).
Let

$$F_R^k(0) = \sum_{\substack{|z|\uparrow \\ z\in \mathbf{B}(0,R)\cap\mathcal{Z}}} \frac{z_k}{|z|^d},$$

be a component of the vector $F_R(0)$, where z_k is the k-th component of z.
Case $1 : d = 2$, for $k \in \{1, 2\}$

$$\begin{aligned}
\varphi_{F_R^k}(t) &= \mathbb{E}[\exp(itF_R^k(0))] \\
&= \exp\Big\{ \int_{\mathbb{R}^2} [\exp(it\frac{z_k}{|z|^2}1_{\{0\le|z|\le R\}}) - 1]dz \Big\} \\
&= \exp\Big\{ \int_0^{2\pi}\int_0^R [\exp(\frac{itr\cos(\theta)}{r^2}) - 1]rdrd\theta \Big\} \qquad \textbf{(polar coordinate)} \\
&= \exp\Big\{ 2\pi \int_0^R \frac{r}{2\pi}\int_0^{2\pi} \exp(itr^{-1}\cos(\theta))d\theta dr - 2\pi\int_0^R rdr \Big\} \\
&= \exp\Big\{ 2\pi \int_0^R \mathbf{J}_0(tr^{-1})rdr - 2\pi\int_0^R rdr \Big\} \\
&= \exp\Big\{ 2\pi\Big[\int_0^1 (\mathbf{J}_0(tr^{-1})r - r)dr + \int_1^R (\mathbf{J}_0(tr^{-1})r - r)dr \Big] \Big\} \\
&= \exp\Big\{ \underbrace{\int_1^\infty \mathbf{J}_0(tu)\frac{2\pi}{u^3}du}_{I_1} - \pi \Big\} \times \exp\Big\{ \int_{1/R}^1 \mathbf{J}_0(tu)\frac{2\pi}{u^3} - \frac{2\pi}{u^3}du \Big\} \qquad \textbf{(change of variable)}
\end{aligned}$$

where $\mathbf{J}_0(x)$ is the Bessel function defined above 4.2.1.
We will show that I_1 is integrable, and $\varphi_{F_R^k}$ goes to 0 as R goes to ∞ .
But, $\mathbf{J}_0(x) \overset{\infty}{\approx} \sqrt{\frac{2}{\pi x}}\cos(x - \frac{\pi}{4})$, so near infinity

$$\Big|\frac{2\pi}{u^3}\mathbf{J}_0(tu)\Big| = \Big|\frac{2\pi}{u^3}\sqrt{\frac{2}{\pi tu}}\cos(tu - \pi/4)\Big| \le \underbrace{\frac{2\sqrt{2\pi t^{-1}}}{u^{7/2}}}_{h(u)},$$

with h integrable so, I_1 is integrable.
Let $C_1 = \exp(I_1 - \pi)$, so we have,

$$\begin{aligned}
\lim_{R\to\infty}\varphi_{F_R^k}(t) &= \lim_{R\to\infty} C_1 \exp\Big\{ 2\pi\int_{1/R}^1 \frac{1}{u^3}(\mathbf{J}_0(tu) - 1)du \Big\} \\
&= C_1 \exp\Big\{ 2\pi\int_0^1 \frac{1}{u^3}\Big[\sum_{m\ge0} \frac{(-1)^m}{m!\Gamma(m+1)}\Big(\frac{tu}{2}\Big)^{2m} - 1\Big]du \Big\} \qquad \textbf{(Taylor expansion)} \\
&= C_1 \exp\Big\{ 2\pi\Big[\int_0^1 \frac{-t^2}{2^2u}du + \int_0^1 \sum_{m\ge2} \frac{(-1)^m u^{2m-3}}{m!(m+1)!2^{2m}}du \Big] \Big\} \\
&= C_1 \exp\Big\{ 2\pi\Big[\int_0^1 \frac{-t^2}{2^2u}du + \sum_{m\ge2}\int_0^1 \underbrace{\frac{(-1)^m u^{2m-3}}{m!(m+1)!2^{2m}}}_{f_m(u)}du \Big] \Big\} \\
&= C_1 \exp\Big\{ 2\pi\Big[\underbrace{\int_0^1 \frac{-t^2}{2^2u}du}_{I_2} + \sum_{m\ge2} \underbrace{\frac{(-1)^m}{m!(m+1)!2^{2m}(2m-2)}}_{a_m} \Big] \Big\}
\end{aligned}$$

First we was able to apply Fubini's theorem to interchange the limit and the sum since,

$$\sum_{m\geq 2}\int_0^1 |f_m(u)|du = \sum_{m\geq 2}\frac{1}{m!(2m-2)(m+1)!2^{2m}},$$

and this sum is convergent by the ratio test. Moreover, by the ratio test we have $\sum a_m$ is also convergent.
Furthermore, I_2 is not intergable and goes to $-\infty$. Thus, $\lim_{R\to\infty}\varphi_{F_R^k}(t)=0$ so, by Levy's theorem (4.2.2) $F_R(0)$ is not convergent.

Case 2 : $d\geq 3$,
$\varphi_{F_R^k}(t)=\mathbb{E}[\exp(itF_R^k(0))]=\exp\{\int_{\mathbb{R}^d}[\exp(\frac{itz_k}{|z|^d}1_{\{0\leq|z|\leq R\}})-1]dz\}$.
Spherical coordinate:
$z_1=r\cos(\theta)$,
$dz=r^{d-1}\sin^{d-2}(\phi_1)....\sin(\phi_{d-2})drd\phi_1...d\phi_{d-1}$.
So we have,

$$\begin{aligned}
\lim_{R\to\infty}\varphi_{F_R^k}(t) &= \lim_{R\to\infty}\exp\Bigg\{\int_0^R\int_0^{2\pi}\int_0^{\pi}...\int_0^{\pi}\Big[\exp(\frac{itr\cos(\phi_1)}{r^d})-1\Big]r^{d-1}\sin^{d-2}(\phi_1)\\
&\qquad ...\sin(\phi_{d-2})d\phi_1...d\phi_{n-2}d\phi_{n-1}dr\Bigg\}\\
&=\exp\left\{K_1\int_0^{\infty}\int_0^{\pi}\Big[\exp(itr^{1-d}\cos(\phi_1))-1\Big]r^{d-1}\sin^{d-2}(\phi_1)d\phi_1dr\right\}\\
&\textbf{(where, } K_1=\int_0^{2\pi}...\int_0^{\pi}\sin^{d-3}(\phi_2)...\sin(\phi_{d-2})d\phi_2...d\phi_{n-1}\geq 0\textbf{)}.\\
&=\exp\left\{K_1\int_0^{1}\int_0^{\pi}\Big[\exp(itr^{1-d}\cos(\phi_1))-1\Big]r^{d-1}\sin^{d-2}(\phi_1)d\phi_1dr\right.\\
&\qquad\left.+K_1\int_1^{\infty}\int_0^{\pi}\Big[\exp(itr^{1-d}\cos(\phi_1))-1\Big]r^{d-1}\sin^{d-2}(\phi_1)d\phi_1dr\right\}\\
&=K_2\exp\left\{K_1\int_1^{\infty}\int_0^{\pi}\Big[\exp(itr^{1-d}\cos(\phi_1))-1\Big]r^{d-1}\sin^{d-2}(\phi_1)d\phi_1dr\right\}\\
&\textbf{(where, } K_2=\exp\left\{K_1\int_0^{1}\int_0^{\pi}\Big[\exp(itr^{1-d}\cos(\phi_1))-1\Big]r^{d-1}\sin^{d-2}(\phi_1)d\phi_1dr\right\}\textbf{)}.\\
&=K_2\exp\left\{K_1\int_0^{\pi}\int_1^{\infty}\Big[\sum_{n\geq 0}\frac{(itr^{-d+1}cos(\phi_1))^n}{n!}-1\Big]r^{d-1}\sin^{d-2}(\phi_1)d\phi_1dr\right\}\\
&=K_2\exp\Bigg\{\underbrace{K_1\int_0^{\pi}\int_1^{\infty}\sum_{n\geq 1}\underbrace{\frac{(itcos(\phi_1))^n}{n!}r^{(1-d)(n-1)}\sin^{d-2}(\phi_1)}_{f_n(r,\phi_1)}d\phi_1dr}_{I_3}\Bigg\}
\end{aligned}$$

Or, $|f_n(r,\phi_1)|\leq\underbrace{\frac{r^{(1-d)(n-1)}}{n!}}_{g_n(r,\phi_1)}$, and $(1-d)(1-n)+1\leq -1$ so,

$$\int_0^{\pi}\int_1^{\infty}g_n(r,\phi_1)drd\phi=\pi\frac{r^{(1-d)(n-1)+1}}{n![(1-d)(1-n)+1]}\Big]_{r=1}^{r=\infty}=\frac{\pi}{n![(d-1)(1-n)-1]}=a_n$$

Applying the ratio test, we get that $\sum a_n$ is convergent. Therefore, by Fubini's theorem I_3 is integralble thus, by Levy's theorem (4.2.2) $F_R(0)$ is convergent. □

4.3 The rearrangement identity

The problem with the previous formulation of the function F appears when we need to differentiate it with respect to x since, x appear in the condition of the summation !
To avoid this problem, we need to consider a new definition for F where x doesn't intervene in the order of the summation.
Let $u, x \in \mathbb{R}^d$, and define

$$G^{\{u\}}(x) = \sum_{z\in\mathcal{Z},\, |z-u|\uparrow} \frac{z-x}{|z-x|^d}$$

(here the condition appear to be an increasing distance from u and not from x);
this sum is defined a.s., and is called **the rearrangement identity**.
Furthermore, for any $x, u, v \in \mathbb{R}^d$ we have that a.s.

$$G^{\{u\}}(x) - G^{\{v\}}(x) = \kappa_d(u-v).$$

(In the original article we were given a sketch of the proof, we aim to write the proof explicitly).

Proof. By a simple computation of $\mathbb{E}[G_R^{\{u\}}(x) - G_R^{\{v\}}(x)]$, where $G_R^{\{u\}}(x)$ is a truncated series of $G^{\{u\}}(x)$ we find that this expectation is equal to $\kappa_d(u-v)$. So, if we show that its variance goes to zero we get that this series converges to its expectation in $\mathbf{L}^2$. Also, we have $(G_R^{\{u\}}(x) - G_R^{\{v\}}(x))$ converges to $(G^{\{u\}}(x) - G^{\{v\}}(x))$ a.s. so, by the uniqueness of the limit (in probability) we get that

$$(G^{\{u\}}(x) - G^{\{v\}}(x)) = \mathbb{E}(G_R^{\{u\}}(x) - G_R^{\{v\}}(x)) = \kappa_d(u-v).$$

First, let's compute the expectation:
Let $\mathcal{N}_{u,x}$ be the random number of star in the ball $\mathbf{B}(u, |u-x|)$ so, $\mathcal{N}_{u,x} \rightsquigarrow \mathcal{P}(\kappa_d|u-x|^d)$.
Let

$$G_R^{\{u\}}(x) = \sum_{z_i\in\mathcal{Z},\, |z_i-u|<R} \frac{z_i-x}{|z_i-x|^d},$$

be a truncated series of $G^{\{u\}}$.
We will use the shell theorem's of Isaac Newton that says that the total gravitational pull on a point x from a uniformly distributed spherical shell of mass M with center u and radius r is equal to 0 if, $r > |u-x|$ and to $\frac{M(u-x)}{|u-x|^d}$ if $r < |u-x|$ (this last fact follows from the harmonicity of the function $x \mapsto \frac{u-x}{|u-x|^d}$).

Claim 4.3.1. Let $\rho_i^{\{u\}} = |u - z_i|$.
By conditioning on $(\rho_i^{\{u\}})_i$, i.e. the distances of the stars from u as was done in the previous section with $u = 0$ 4.2, we get

$$\mathbb{E}[G_R^{\{u\}}(x) \,|\, \mathcal{N}_{u,x}, (\rho_i^{\{u\}})_i] = \mathcal{N}_{u,x}\frac{u-x}{|u-x|^d}.$$

Proof of claim (4.3.1) Let R be large enough such that $R > |u-x|$, and consider

$$G_R^{\{u\}} = \sum_{\substack{z_i\in\mathcal{Z}\\ |z_i-u|<R}} \frac{z_i-x}{|z_i-x|^d},$$

the truncated series of $G^{\{u\}}$. By conditioning on the $(\rho_i^{\{u\}})_i$ thinking of them as deterministic (as was done in the previous proof 4.2 with $\rho_i^{\{u\}} = |u - z_i|$), we have :

$$\begin{aligned}
\mathbb{E}[G_R^{\{u\}}(x) \,|\, \mathcal{N}_{u,x},\, (\rho_i^{\{u\}})_i] &= \mathbb{E}\Big[\sum_{\substack{z_i \in \mathcal{Z} \\ |u-z_i|<R}} \frac{z_i - x}{|u - z_i|^d} \,|\, \mathcal{N}_{u,x},\, (\rho_i^{\{u\}})_i \Big] \\
&= \mathbb{E}\Big[\Big(\sum_{\substack{z_i \in \mathcal{Z} \\ |u-z_i|<|u-x|}} \frac{z_i - x}{|u - z_i|^d} + \sum_{\substack{z_i \in \mathcal{Z} \\ |u-z_i|>|u-x|}} \frac{z_i - x}{|u - z_i|^d} \Big) \,|\, \mathcal{N}_{u,x},\, (\rho_i^{\{u\}})_i \Big] \\
&= \mathbb{E}\Big[\Big(\sum_{\substack{z_i \in S(0,\rho_i^{\{u\}}) \cap \mathcal{Z} \\ \rho_i^{\{u\}}<|u-x|}} \frac{z_i - x}{|u - z_i|^d} + \sum_{z_i \in B^c(0,|u-x|) \cap \mathcal{Z}} \frac{z_i - x}{|u - z_i|^d} \Big) \,|\, \mathcal{N}_{u,x},\, (\rho_i^{\{u\}})_i \Big] \\
&= \mathbb{E}\Big[\sum_{\substack{z_i \in S(0,\rho_i^{\{u\}}) \cap \mathcal{Z} \\ \rho_i^{\{u\}}<|u-x|}} \frac{z_i - x}{|u - z_i|^d} \,|\, \mathcal{N}_{u,x},\, (\rho_i^{\{u\}})_i \Big] \qquad \textbf{(using the shell theorem)} \\
&= \frac{u-x}{|u-x|^d} \mathbb{E}\Big[\sum_{\substack{z_i \in S(0,\rho_i^{\{u\}}) \cap \mathcal{Z} \\ \rho_i^{\{u\}}<|u-x|}} |\, \mathcal{N}_{u,x},\, (\rho_i^{\{u\}})_i \Big] \\
&= \frac{u-x}{|u-x|^d} \mathbb{E}[\mathcal{N}_{u,x} \,|\, \mathcal{N}_{u,x}] \\
&= \frac{u-x}{|u-x|^d} \mathcal{N}_{u,x}.
\end{aligned}$$

◇

And so we have,

$$\begin{aligned}
\mathbb{E}[G_R^{\{u\}}(x)] = \mathbb{E}\big[\mathbb{E}[G_R^{\{u\}}(x) \,|\, \mathcal{N}_{u,x},\, (\rho_i^{\{u\}})_i]\big] &= \mathbb{E}[\frac{u-x}{|u-x|^d} \mathcal{N}_{u,x}] \\
&= \frac{u-x}{|u-x|^d} \mathbb{E}[\mathcal{N}_{u,x}] = \kappa_d |u-x|^d \frac{u-x}{|u-x|^d} \\
&= \kappa_d (u-x).
\end{aligned}$$

Let $X_R = G_R^{\{u\}}(x) - G_R^{\{v\}}(x) = \sum_{z \in A_R \cap \mathcal{Z}} \frac{z-x}{|z-x|^d} - \sum_{z \in B_R \cap \mathcal{Z}} \frac{z-x}{|z-x|^d}$, where $A_R = \mathbf{B}(u,R)/\mathbf{B}(v,R)$ and $B_R = \mathbf{B}(v,R)/\mathbf{B}(u,R)$, and $X = (G^{\{u\}}(x) - G^{\{v\}}(x))$. Note that:

$$\left.\begin{array}{l} G_R^{\{u\}}(x) \xrightarrow[R\to\infty]{} G^{\{u\}}(x) \;\; a.s \\ G_R^{\{v\}}(x) \xrightarrow[R\to\infty]{} G^{\{v\}}(x) \;\; a.s \end{array}\right\} \Rightarrow X_R \xrightarrow[R\to\infty]{} X \;\; a.s.$$

Partition the set $\mathbf{B}(u,R) \triangle \mathbf{B}(v,R)$ into $\mathcal{O}(R^{d-1})$ disjoints sets $(E_j)_j$ of Lebesgue measure $\mathcal{O}(1)$ s.t. E_j is in A_R or B_R (figure (2)).
Let

$$Y_j = \sum_{z_i \in E_j} \frac{z_i - x}{|z_i - x|^d}$$

be the contribution to the force from stars in E_j;

Claim 4.3.2. The Y_j's are independents, each has variance bounded from above by $\mathcal{O}(1/R^{2d-2})$, and we have $X_R = \sum_j \pm Y_j$.

Proof of claim (4.3.2) Take R large enough s.t. $\mathbf{B}(u,R) \triangle \mathbf{B}(v,R) \subset \mathbb{R}^d \setminus \mathbf{B}(x,R/2)$, so for $z_i \in (\mathbf{B}(u,R) \triangle \mathbf{B}(v,R)) \cap \mathcal{Z}$, we have $|z_i - x| > R/2$.

$$\begin{aligned}
V(Y_j) &= \mathbb{E}[Y_j Y_j^T] - (\mathbb{E}[Y_j])^2 \\
&\leq \mathbb{E}[Y_j Y_j^T] \\
&= \mathbb{E}[\mathbb{E}[Y_j Y_j^T \,|\, Card(E_j \cap \mathcal{Z})]] \\
&= \mathbb{E}[\mathbb{E}[\sum_{z_i \in E_j} \frac{(z_i - x)(z_i - x)^T}{|z_i - x|^{2d}} \,|\, Card(E_j \cap \mathcal{Z})]] \\
&= \mathbb{E}[\mathbb{E}[\sum_{z_i \in E_j} |z_i - x|^{2-2d} \,|\, Card(E_j \cap \mathcal{Z})]] \\
&\leq \mathbb{E}[\mathbb{E}[\sum_{z_i \in E_j} \frac{2^{2d-2}}{R^{2d-2}} \,|\, Card(E_j \cap \mathcal{Z})]] \\
&= \frac{2^{2d-2}}{R^{2d-2}} \mathbb{E}[\mathbb{E}[Card(E_j \cap \mathcal{Z}) \,|\, Card(E_j \cap \mathcal{Z})]] \\
&= \frac{2^{2d-2}}{R^{2d-2}} [\mathbb{E}[Card(E_j \cap \mathcal{Z})]] \\
&= \mathcal{O}(R^{-2d+2})
\end{aligned}$$

Where $\mathcal{O}(R^{-2d+2})$ is independent of j. ◇

$$\begin{aligned}
V(X_R) &= V(\sum_j \pm Y_j) \\
&= \sum_j V(\pm Y_j) \qquad \textit{(as } Y_j\textit{'s are independents)} \\
&\leq \sum_j \mathcal{O}(R^{-2d+2}) \\
&= \mathcal{O}(R^{-2d+2}) \mathcal{O}(R^{d-1}) \quad \textit{(we have } \mathcal{O}(R^{d-1}) \textit{ disjoint } (E_j)_j\textit{)} \\
&= \mathcal{O}(R^{-d+1}) \xrightarrow{R \to \infty} 0.
\end{aligned}$$

Therefore, $X_R - \mathbb{E}(X_R) \xrightarrow{R \to \infty} 0$ in $\mathbf{L}_2$ but, $\mathbb{E}(X_R) = \kappa_d(u - v)$ which doesn't depends on R.
Thus, $X_R \xrightarrow{R \to \infty} \kappa_d(u - v)$ in $\mathbf{L}_2$, so in probability.
But, $X_R \xrightarrow{R \to \infty} X$ a.s., so in probability.
By uniqueness of limit, we get $X = \kappa_d(u - v)$ which end the proof.

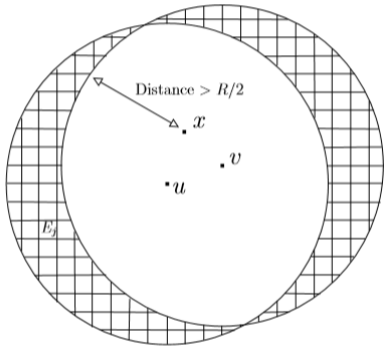


Figure 2: The balls $\mathbf{B}(u,R)$ and $\mathbf{B}(v,R)$ and the sets E_j. (From [2]).

□

4.4 Alternative formulation of F

Now, by choosing $u = 0$ and adding the correcting term we get an alternative expression for the force function $F(x)$ that does not involve a different order of simulation at every point x.
For any $x \in \mathbb{R}^d \setminus \mathcal{Z}$ we have

$$F(x) = \sum_{z \in \mathcal{Z}, |z|\uparrow} \frac{z - x}{|z - x|^d} + \kappa_d x. \tag{4.4.3}$$

One of the most interesting and surprising results is that a.s. all the basins of attraction have a volume equal to 1.

4.5 Fairness of the allocation

In this section we will give an idea about why each basin of attraction has an area equal to one. This proof is based on the divergence theorem, but it requires the assumption that each basin of attraction is bounded by smooth curves, to see the formal proof, we refer to [2].

Proof. Assume that the basins are bounded by finitely many smooth curves.
Take a basin of attraction $\mathcal{B}(z)$, and a point $x \in \partial\mathcal{B}(z)$. If n is the outward-pointing normal vector at x then, by definition of a basin of attraction we have $F(x).n = 0$ since the way the boundaries are constructed gives that the force is balanced on the boundaries. Because, if the force will point into the basin of attraction then, this won't be in the boundary, it will be attracted to the basin. Therefore, the normal vector is orthogonal to the force of attraction.
Thus, the oriented surface integral

$$\int_{\partial\mathcal{B}(z)} F(x).n dS = 0.$$

Furthermore,

$$\operatorname{div}(F) = d\kappa_d - d\kappa_d \sum_{i=1}^{\infty} \delta_{z_i}.$$

But, by definition of a basin of attraction, we just have one star inside it (unique singularity) hence,

$$\operatorname{div}(F) = d\kappa_d - d\kappa_d \delta_z.$$

By the divergence theorem's:

$$0 = \int_{\partial\mathcal{B}(z)} F(x).n dS = \int_{\mathcal{B}(z)} \operatorname{div}(F) dx = d\kappa_d \operatorname{Vol}(\mathcal{B}(z)) - d\kappa_d.$$

Therefore, $\operatorname{Vol}(\mathcal{B}(z)) = 1$. □

5 Zeros of complex analytic functions and the Laplacian of the Logarithm

In this section, we talk about complex analytic functions and we give the proof that

$$dn_f(z) = \frac{1}{2\pi} \Delta \log |f(z)|,$$

(where n_f is the counting measure on the zero set of a complex analytic function f) since we need it to compute the Laplace operator in a next section; also we give some definitions needed in our work.

5.1 Gaussian analytic function (G.A.F.)

Definition 5.1.1. "Complex Analytic function"
An analytic function is a function that is locally given by a convergent power series.
A complex function is said to be analytic on a region R if it is complex differentiable at every point in R. The terms holomorphic function, differentiable function, and complex differentiable function are sometimes used interchangeably with "analytic function". Many mathematicians prefer the term "holomorphic function" (or "holomorphic map") to "analytic function", while "analytic" appears to be in widespread use among physicists, engineers, and in some older texts. The word "analytic" is defined in a broader sense to denote any function (real, complex, or of more general type) that can be written as a convergent power series in a neighbourhood of each point in its domain.
A complex function that is analytic at all finite points of the complex plane is said to be entire.

N.B. Complex analytic functions exhibit properties that do not hold generally for real analytic functions.

Definition 5.1.2. "Complex Gaussian"
A standard complex Gaussian is a complex-valued random variable with probability density $\frac{1}{\pi}e^{-|z|^2}$ with respect to the Lebesgue measure on the complex plane. In other words, one may define it as $X + iY$ where X and Y are random variables i.i.d. $\rightsquigarrow \mathcal{N}(0, \frac{1}{2})$, but we advocate thinking of it as a single entity, if not to think of a real Gaussian as merely the real part of a complex Gaussian.

Indeed, one encounters the complex Gaussian variable in basic probability courses, for instance in computing the normalizing constant for the density $e^{-x^2/2}$ on the line (by computing the normalizing constant for a complex Gaussian and taking square roots); and also in generating a random normal on the computer (by generating a complex Gaussian and taking the real part).
The complex Gaussian is sometimes easier to work with because it can be represented as a pair of independent random variables in two coordinate systems, cartesian as well as polar. At a higher level in the theory of random analytic functions and random matrix theory, it is again true that many more exact computations are possible when we use complex Gaussian coefficients (or entries) than when real Gaussians are used.

Definition 5.1.3. "Gaussian analytic function (G.A.F.)"
A Gaussian analytic function (denoted G.A.F.), is a linear combination

$$f(z) = \sum_{k\geq 0} \xi_k f_k(z)$$

of analytic functions $f_k : G \subset \mathbb{C} \longrightarrow \mathbb{C}$ such that

$$\sum_{k\geq 0} |f_k(z)|^2 < \infty$$

with independent standard complex Gaussian random coefficients ξ_k.

5.2 Green's function of Laplace operator

Definition 5.2.1. "Green's function"
A Green function $G(x, s)$ of a linear differential operator $L = L(x)$ acting on distributions over of the Euclidean space $\mathbb{R}^d$ at a point s, is any solution of

$$LG(x, s) = \delta(s - x) \tag{5.2.4}$$

where δ is the dirac delta function.

This propriety of Green's function can be exploited to solve differential equations of the form

$$Lu(x) = f(x).$$

Also, Green's functions in general are distributions not necessarily functions of a real variable. Loosely speaking if such a Green function G can be found for the operator L then, if we multiply (5.2.4) for the Green's function by $f(s)$ and integrate with respect to s, we obtain

$$\int LG(x,s)f(s)ds = \int \delta(x-s)f(s)ds = f(x) \tag{5.2.5}$$

because, the operator $L = L(x)$ is linear, and acts only on the variable x.

Lemma 5.2.1. In dimension $d = 2$, the Green's function of the Laplace operator $L = \Delta$, is :

$$G(x,s) = \frac{1}{2\pi}\log|s-x|.$$

5.3 Zeros of analytic functions

Proposition 5.3.1. Let f be an analytic function and n_f be the counting measure (with multiplicities) on $f^{-1}\{0\}$.
Then,

$$dn_f(z) = \frac{1}{2\pi}\Delta\log|f(z)|. \tag{5.3.6}$$

Here the Laplacian on the right hand side should be interpreted in the distributional sense. In other words, the meaning of (5.3.6) is just that for any smooth function ϕ compactly supported, we have

$$\int \phi(z)dn_f(z) = \int \Delta\phi(z)\frac{1}{2\pi}\log|f(z)|dm(z).$$

We give the proof of this proposition since we'll need it in our construction in the next sections to verify the computation of the Laplace of the potential function that we will define.

Proof. Let ϕ be a smooth function compactly supported, and let Λ be the compact support of ϕ. We need to show that:

$$\int_\Lambda \phi(z)dn_f(z) = \int_\Lambda \Delta\phi(z)\frac{1}{2\pi}\log|f(z)|dm(z).$$

To see this write $f(z) = g(z)\prod_k (z-\alpha_k)^{m_k}$, where α_k are the zeros of f in the support of ϕ, and g is an analytic function with non zeros in the support of ϕ.
As ϕ is compactly supported, there are only finitely many α_k so,

$$\log|f(z)| = \log|g(z)| + \sum_k m_k \log|z-\alpha_k|$$

Moreover, $\Delta\log|g|$ is identically zero on the support of ϕ because, $\log|g|$ is locally the real part of an analytic function.
By lemma 5.2.1, $\frac{1}{2\pi}\log|z-\alpha_k| = G(\alpha_k, z)$ the Green's function for the operator $L = \Delta$ the Laplacien in the plane, so

$$\int_\Lambda \Delta\phi(z)\log|z-\alpha_k|dm(z) = \phi(\alpha_k).$$

Thus,

$$\begin{aligned}\int_\Lambda \Delta\phi(z)\frac{1}{2\pi}\log|f(z)|dm(z) &= \int_\Lambda \Delta\phi(z)\Big[\frac{1}{2\pi}\log|g(z)| + \frac{1}{2\pi}\sum_k m_k \log|z-\alpha_k|\Big] \\ &= \frac{1}{2\pi}\int_\Lambda \Delta\phi(z)\log|g(z)|dm(z) + \sum_k m_k \int_\Lambda \phi(z)\frac{1}{2\pi}\Delta\log|z-\alpha_k|dm(z) \\ &= \sum_k m_k\phi(\alpha_k) \\ &= \int_\Lambda \phi(z)dn_f(z).\end{aligned}$$

□

6 Interpretation

One can ask about the way of construction and the properties to be provided, to build a fair gravitational allocation.
Hence in this section, we try to analyze the previous gravitational allocation 4, to fix some ideas, and we hope to be able to construct an allocation rule to another point process.
Thinking deeply about the construction of the gravitational allocations, it's important to know why the authors have chosen this force functions 4.4?
If we aim to construct a new allocation what function should we consider? What are the proprieties to be satisfied to have equal area?
Back to the allocation scheme defined before 4:
The force function was taken to be:

$$F(x) = \sum_{z\in\mathcal{Z},|z|\uparrow} \frac{z-x}{|z-x|^d} + \kappa_d x,$$

while its the divergence is:

$$\mathrm{div}(F) = -d\kappa_d \sum_{z\in\mathcal{Z}} \delta_z + d\kappa_d,$$

and the gradient curves was defined as solution of the differential equation:

$$\frac{dY_x(t)}{dt} = F(Y_x(t)), \quad Y_x(0) = x.$$

6.1 Potential function

First, we remark that the divergence of the force function F is positive on $\mathbb{R}^d \setminus \mathcal{Z}$. Thus, if we suppose that F is the negative of the gradient of a potential function u, we figure out that the potential functions is a super-harmonic functions on $\mathbb{R}^d \setminus \mathcal{Z}$.
So we can image this scheme:

- A potential function u super-harmonic on $\mathbb{R}^d \setminus \mathcal{Z}$, i.e.

  $$\Delta u(x) \leq 0,$$

  for all $x \in \mathbb{R}^d \setminus \mathcal{Z}$.

- A force function F s.t.

  $$F(x) = -\nabla u(x)$$

- $$\Delta u = -\mathrm{div}(F)$$

- The gradient curves $Y_x(t)$ are the unique solution of the differential equation

$$\frac{dY_x(t)}{dt} = F(Y_x(t)) = -\nabla u(Y_x(t)), \quad Y_x(0) = x$$

on a maximal domain $]-\infty, \tau_x[$, where $0 < \tau_x \leq \infty$.

As we know, a minimum of a super-harmonic function can not be achieved in the interior of its domain unless the function is constant. This is the so-called **minimum principle** [20], which implies that the potential functions have no local minimum other than the points of $\mathcal{Z}$.
Moreover, we orient the gradient curves in the direction of decay of u (this is the reason for the choice of the minus sign in the differential equations).
Therefore, for a typical starting point, the gradient will flow down to a point of the considered point process.
Thus the harmonicity is a necessary condition, to be satisfied by the potential function, that we consider to define a gravitational allocation.

6.2 Volume of the basins

Another important property that we must pay attention, before defining the potential function (or the force function), is the intensity of the considered point process.
We cannot hope to allocate the whole space (up to a set of Lebesgue measures equal to zero), to a point process of intensity $\frac{1}{c}$, if the allocation rule gives to each star a territory having an area different then c!
So, if we hope to allocate all the space (up to a Lebesgue measure set equal to zero), we must construct an allocation which gives each star, a basin of attraction of area equal to the inverse of the intensity of the point process. Hence depending on the intensity of the point process, we should modify the potential function, or the force function (if we didn't define the potential function) to provide the above condition.
To this aim, if we suppose that the basins of attraction of the stars are bounded by finitely smooth curves, and we apply the divergence theorem to the force function we can see what really affect the volume of the basins!
Note that this doesn't give us a proof of the fairness of the allocation, but a first method to estimate the volume.

6.3 Point process

The below figures show a simulation of a piece of the gravitational allocation when applied to a finite number of points chosen uniformly from a square, i.e a finite approximation of Poisson process on the plane (on the left), and (on the right) to a finite number of zeros of (G.E.F.) :

$$f(z) := \sum_{k=0}^{\infty} \xi_k \frac{z^k}{\sqrt{k!}} \qquad \text{(G.E.F.)}$$

where, $\xi_0, \xi_1, ...$ are Gaussian i.i.d. complex random variables.
Note that (G.E.F.) is a Gaussian analytic function 5.1.3 whose set of zeros form a point process $\mathcal{Z}_f$, with first intensity equal to π^{-1}.

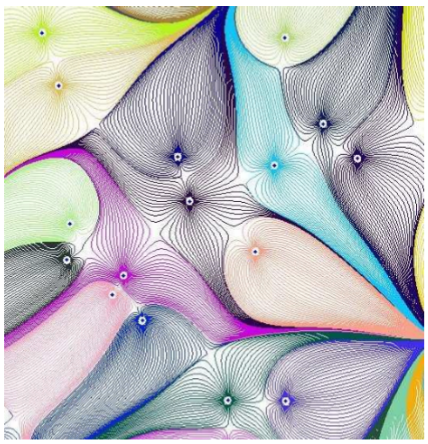


Figure 3: Gravitational allocation to the Poisson point process. [4]

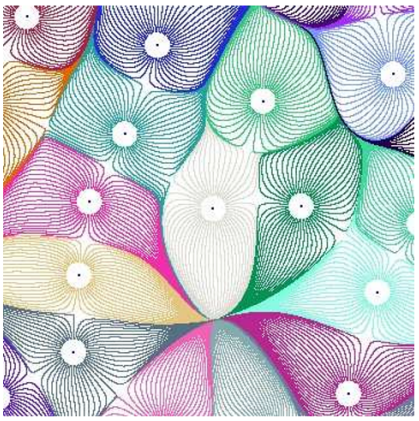


Figure 4: Gravitational allocation to the zeros of (G.E.F.). [9]

The zeros of the Gaussian analytic functions (G.A.F.) exhibit some local repulsion [4]. Due to this propriety of repulsion, they have less clustering of points than the Poisson point process.
Visibly the basins in the above figures are more elongated in the case of Poisson compared to the case of the zeros of the (G.E.F.). This may be a consequence of the difference in the configuration between the Poisson point process, and the zeros of the G.A.F. (in our case the zeros of the Gaussian analytic function (G.E.F.)).
Thus, the Gravitational allocation seems to be more economic for the zeros of the G.A.F., than for the point of the Poisson process.

The zeros of the G.A.F. share with Determinantal point processes, the similar local repulsion but, this propriety doesn't hold globally since a Determinantal point process tends to repeal in all direction, while the zeros of the G.A.F. repeal-only in a short distance. This motivates us to look forward to construct a gravitational allocation to a Determinantal point process, hoping to get a more localized, economic allocation.
In the what follows, we define a gravitational allocation from Lebesgue to the Ginibre point process which is one of the Determinantal point processes.

7 Gravitational allocation from Lebesgue to the Ginibre point process

In this section, we construct a gravitational allocation to the Ginibre point process. This point process is important being one of the main examples of Determinantal point processes on the complex plane. It forms a recurring model in stochastic matrix theory as well as in practical applications. Since its introduction in random matrix theory, the Ginibre point process has also been used to model random phenomena where repulsion is observed. Many difficulties arise while working with the Ginibre point process in the field of gravitational allocations.
We'll start by some definitions.

7.1 Determinantal point process

Determinantal point processes are distributions over configurations of points that encode diversity through a kernel $\mathbb{K}$. They were first introduced by Macchi (1975) as models for beams of fermions, and they have since found applications in fields as diverse as probability, statistical physics, and machine learning.

Definition 7.1.1. "Determinantal point process (DPP)"
Let Λ be a locally compact Polish space, and ν a Radon measure on Λ.
Let $\mathbb{K}(x, y) : \Lambda^2 \to \mathbb{C}$ be a measurable function.
A point process $\mathcal{D}$ on Λ is said to be a **Determinantal point process** with kernel $\mathbb{K}$, if it is simple, and its joint intensities with respect to the measure ν satisfy

$$\rho_k(x_1, x_2, ..., x_k) = \det(\mathbb{K}(x_i, x_j))_{1 \leq i,j, \leq k},$$

for every $k \geq 1$ and $x_1, ..., x_k \in \Lambda$.

In the machine learning field, in general a finite version of the Determinantal point is used, by setting $\Lambda = \{1, ..., N\}$ and $\nu = \sum_{i=1}^{N} \delta_i$. In this context, the kernel function becomes an $N \times N$ matrix $\mathbb{K}$, and the correlation functions refer to inclusion probabilities. Determinantal point process are then defined as $\mathcal{D} \rightsquigarrow DPP(\mathbb{K})$ if :

$$\mathbb{P}[S \subset \mathcal{D}] = \det(\mathbb{K}_S), \qquad \forall S \in \Lambda,$$

where $\mathbb{K}_S$ denotes the submatrix of $\mathbb{K}$ formed by the rows, and columns indexed by S.
The kernel matrix $\mathbb{K}$ is commonly assumed to be real-symmetric, where the existence and uniqueness of the DPP is equivalent to the condition that the eigenvalues of $\mathbb{K}$ lie in $[0, 1]$.

7.2 Ginibre point process

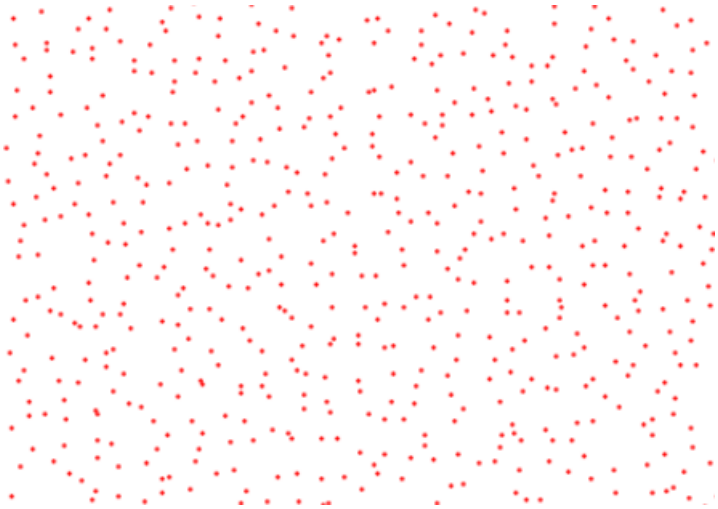


Figure 5: Ginibre point process.

In 1965 J.Ginibre introduced three ensembles of random matrices whose entries were respectively chosen with Gaussian density from $\mathbb{R}, \mathbb{C}$ and Hamilton's quaternion. Here we will work with the complex case; this example is an important one as the points in the complex plane are similar to, yet different from the zeros of Gaussian analytic functions [8]. Moreover, the Ginibre point process is the natural extension of the Dyson point process to the complex plane. As such, and as explained in [11], it models the positions of charges of a two-dimensional Coulomb gas in a harmonic oscillator potential, at a temperature corresponding to $\beta = 2$. It should be noted that the Dyson model is a Determinantal point process on $\mathbb{R}$ which is of central importance, as it appears as the bulk-scaling limit of a large class of Determinantal point processes. More recently, different authors have used the Ginibre point process to model phenomena arising in networking. Indeed, this particular model has many advantages with regard to its applications. It is indeed invariant with respect to rotations and translations. Furthermore, the electrostatic repulsion between particles seems to be fitting for

many applications.
The main problem that arises in practice is that although the eigenvalues of matrices in the GUE ensemble form a Ginibre point process, these eigenvalues are *not compactly supported*, although, after renormalization, they tend to be compactly supported as N tends to infinity (this is known as the circular law in stochastic matrix theory). Furthermore, truncating to a naturally compact and letting N tend to infinity is not the most efficient way to proceed, even though this operation preserves the Determinantal property of the point process.

Our aim in this section is to construct a fair gravitational allocation rule from Lebesgue to the Ginibre point process, so we start by giving a general way to construct a gravitational allocation to the infinite Ginibre point process and then we construct a gravitational allocation to the Truncated Ginibre point process in a compact space.

Definition 7.2.1. " Ginibre point process"
The Ginibre point process, denoted by $\mathcal{G}$ in the remainder of this paper, is defined as the Determinantal point process on $\mathbb{C}$ with integral kernel

$$\mathbb{K}(z_1, z_2) = \frac{1}{\pi} e^{z_1 \bar{z}_2} e^{-\frac{1}{2}(|z_1|^2 + |z_2|^2)}, \qquad z_1\,,\, z_2\ \in \mathbb{C},$$

with respect to $\mu := dl(z)$, the Lebesgue measure on $\mathbb{C}$ (i.e. $dl(z) = dxdy$, where $z = x + iy$).
It can be naturally decomposed as

$$\mathbb{K}(z_1, z_2) = \sum_{n \geq 0} \phi_n(z_1)\bar{\phi}_n(z_2), \qquad z_1\,,\, z_2\ \in \mathbb{C}, \tag{7.2.7}$$

where $\phi_n(z) := \frac{1}{\sqrt{\pi n!}} e^{-\frac{1}{2}|z|^2} z^n$, for $n \in \mathbb{N}$ and $z \in \mathbb{C}$.
The Ginibre point process $\mathcal{G}$ verifies the following basic proprieties:

Proposition 7.2.1. The Ginibre point process $\mathcal{G}$ satisfies the following: [14]

1. $\mathcal{G}$ is ergodic with respect to the translations on the plane.
2. $\mathcal{G}$ is isotropic.
3. $\mathcal{G}(\mathbb{C}) = +\infty$ almost surely, i.e. the Ginibre point process has an infinite number of points almost surely.

Since $\mathcal{G}$ has an infinite number of points almost surely, it is impossible to simulate it directly. So we'll focus our work on the finite Ginibre point process.

In what follow we will construct the gravitational allocation to the Ginibre point process, then we'll look forward the gravitational allocation to a Truncated Ginibre point process by modifying the kernel $\mathbb{K}$ in order to obtain a version of the Ginibre point process in a compact set.

7.3 Gravitational allocation from Lebesgue to the Ginibre point process

In this section, we'll construct transportation of the Lebesgue measure to the Ginibre point process.
We know from [14] that the particle density of the Ginibre point process is constant and, is equal to $1/\pi$ and from [5] that for a realization of Ginibre, there exists a Gaussian analytic function with randomized covariance kernel [5] such that the roots of this function are the points of the Ginibre. This will give us a way to construct the gravitational allocation to the Ginibre point process.

Let $\mathcal{G} = \{z_1, z_2, ..., z_n, ...\}$ be a realisation of the Ginibre ensemble in the complex plane $\mathbb{C}$, and let g be the Gaussian analytic function with randomized covariance kernel such that $\mathcal{G} = g^{-1}\{0\}$. If we denote by n_f the counting measure (with multiplicities) on $g^{-1}\{0\}$ then,

$$dn_f(z) = \frac{1}{2\pi}\Delta \log|g(z)|,$$

where the Laplacian on the right hand side should be interpreted in the distributional sense (see (5.3.6)).
For any $x \in \mathbb{C}$ we define a potential function u to be

$$u(x) := \log|g(x)| - \frac{1}{2}|x|^2.$$

For $x \in \mathbb{C} \setminus \mathcal{G}$, let

$$-\nabla u(x) = F(x),$$

be the force on x arising from the potential u. This force represent the speed rather than being proportional to the acceleration as in Newtonian gravitation.
For any $x \in \mathbb{C}$, consider the gravitational flow curve $Y_x(t)$ to be the solution of the differential equation,

$$\frac{dY_x}{dt}(t) = F(Y_x(t)),\ Y_x(0) = x.$$

For each x, the curve $Y_x(t)$ can be defined over some maximal domain $]-\infty, \tau_x[$, where $0 < \tau_x \leq \infty$. We than define the gravitational allocation to be the allocation rule given by,

$$\psi(x) = \begin{cases} z & If \lim\limits_{t\to\tau_x} Y_x(t) = z \text{ and } z \in \mathcal{G}, \\ \infty & Otherwise. \end{cases}$$

For $z \in \mathcal{G}$, let

$$\mathcal{B}(z) = \{x \in \mathbb{R}^d \setminus \mathcal{G},\ Y_x(t) \text{ ends at } z\} \cup \{z\},$$

be the basin associated to the star z.

7.3.1 Conditional Fairness of the allocation

Proposition 7.3.1. If each basin is bounded by finitely many smooth curves, then the function ψ given above (7.3) define an allocation of the Lebesgue measure to $\mathcal{G}$ that divides the lands into equal sets of area π.

Proof. Assume that the basins defined by (7.3) are bounded by finitely many smooth curves.
Let $z \in \mathcal{G}$, and take $x \in \partial B(z)$.
If n is the outward-facing unit normal vector to $\partial B(z)$ at x, then by definition of basin $F(x).n = 0$ since, the way the boundaries are constructed gives that the force is balanced on the boundaries because if the force will point into the basin then this won't be in the boundary, it'll be attracted to the basin. Therefore, the normal is orthogonal to the force of attraction.
Thus, the oriented surface integral

$$\int_{\partial B(z)} F(x).ndS = 0.$$

Furthermore,

$$\text{div}(F) = -\Delta(u) = 2 - 2\pi\sum_{z\in\mathcal{G}} \delta_z. \tag{7.3.8}$$

But, by definition of basin of attraction we just have one star (unique singularity) hence,

$$\mathrm{div}(F) = 2 - 2\pi\delta_z.$$

By the divergence theorem and using (7.3.8)

$$0 = \int_{\partial B(z)} F(x).ndS = \int_{B(z)} \mathrm{div}(F)dx = 2\mathrm{Vol}(B(z)) - 2\pi.$$

Therefore, $\mathrm{Vol}(B(z)) = \pi$. □

In this section we have constructed a gravitational allocation to the Ginibre point process, and we have proved that this construction could be a fair allocation (in the sense that the cells have equal area), if the condition of having each basin bounded by finitely many smooth gradient curves is satisfied.
Our construction was based on the idea of considering that each sample of the Ginibre point process could be considered as the zeros of a random analytic function.
Moreover, for a point process, one quantity of interest is decay rate of the *hole probability,* that is the probability that a disk of radius r contains no points, as $r \longrightarrow \infty$. One can consider this quantity as a rough measure of the mutual repulsion (or "*rigidity*") in the process (see [4] section 7). In the article [13], the author state that both Ginibre ensemble, and the zeros of the G.A.F. [9] the hole probability decays like $\exp(-cr^4(1 + O(1)))$, while for the Poisson point process, which exhibits no rigidity, the decay rate is $\exp(-cr^2)$.
This fact about the decay gives us an idea about how economic this allocation rule will be. As the probability of the Ginibre point process decays like the zeros of the G.A.F. this implies that our construction may be as economic as the allocation to the zeros of the G.A.F. [9], and more economic than the allocation to the Poisson point process [2].
The problem of this construction is that it couldn't be simulated in practice!
In fact, to simulate any allocation from Lebesgue to a point process $\mathcal{Z}$, we need to apply this construction to a finite number of point say N. So the first problem that arises is two-fold, first finding a finite version of the considered point process $\mathcal{Z}^N$ such that $\mathcal{Z}^N$ converges to $\mathcal{Z}$ in distribution, and finding a way to sample $\mathcal{Z}^N$.
Another problems arise, related to the goal of our construction, for example when the particle density of the finite point process $\mathcal{Z}^N$ is not constant, or the support of $\mathcal{Z}^N$ is the whole space.
As our goal is to construct a gravitational allocation from Lebesgue to the Ginibre point process, we need first to find a finite version of the Ginibre point process, which we can control the number of points that fall in a given compact space, that converges to the Ginibre point process and that we can sample in practice. To that end, we will work with the Truncated Ginibre Point Process.

In what follows, we give a modification of the Ginibre point process mentioned in [14], in order to obtain a Determinantal point process more suited for simulation. We also give a method to simulate the Truncated Ginibre point Process. Moreover, we construct a transportation by gradient flow from Lebesgue to the Truncated Ginibre point process and we give an algorithm to simulate it. First, we will introduce the Truncated Ginibre point process.

7.4 Truncated Ginibre point process

The idea is to consider the truncated Ginibre kernel, defined for $N \in \mathbb{N}^*$ by

$$\mathbb{K}^N(z_1, z_2) = \sum_{n=0}^{N-1} \phi_n(z_1)\bar{\phi}_n(z_2), \qquad z_1\,,\, z_2 \in \mathbb{C}. \tag{7.4.9}$$

which is in fact a truncation of the sum in (7.2.7). Additionally, we call $\mathcal{G}^N$ the associated Determinantal point process.
We remark that $\mathcal{G}^N$ converges weakly to $\mathcal{G}$ as N goes to infinity, and $\mathcal{G}^N$ has N points almost surely (for the proof see [14]). Physically, $\mathcal{G}^N$ is the distribution of N polarized electrons in a perpendicular magnetic field, filling the N lowest Landau levels, as is remarked in [15]. It's also known that the radii (in the complex plane) of the points of $\mathcal{G}^N$ have the same distribution as independent gamma random variables.

Simulation of a truncated Ginibre point process with kernel (7.4.9):

For this specific case, there is in fact a natural way of simulating the Ginibre process. Indeed, the eigenvalues of an $N \times N$ hermitian matrix with complex Gaussian entries are distributed according to $\mathcal{G}^N$, more precisely in [5] they give the concept of a finite Ginibre point process as follow:

Definition 7.4.1. "Finite Ginibre point Process"
Let M be an $N \times N$ matrix with i.i.d. standard complex Gaussian entries.
Then, the eigenvalues of M form a Determinantal point process on the complex plane with kernel

$$\mathbb{K}^N(z, w) = \sum_{k=0}^{N-1} \frac{(z\bar{w})^k}{k!} \tag{7.4.10}$$

with respect to the background measure $\frac{1}{\pi}e^{-|z|^2}dl(z)$. This process is called **"The finite Ginibre point process"**. The spectral density of the Ginibre ensemble is described by the Griko circular law according to which the eigenvalue distribution for N large is, in the leading order, uniform in a disk about the origin in the complex plane.

Simulation:

In practice we can obtain N point of the Ginibre point process with kernel (7.4.10) following this way:
Simulate a matrix A of i.i.d. $N \times N$ entries, s.t. $A \rightsquigarrow \frac{1}{\sqrt{2}}(\mathcal{N}(0,1) + i\mathcal{N}(0,1))$. Then find the eigenvalues of A which represent a realisation of N point of the Ginibre ensemble with kernel (7.4.10) (i.e. they are distributed according $\mathcal{G}^N$). *The library DPPy in python give also a very quick simulation of N point of the Ginibre following the above idea.* This is by far the most efficient way of simulating the truncated Ginibre process.

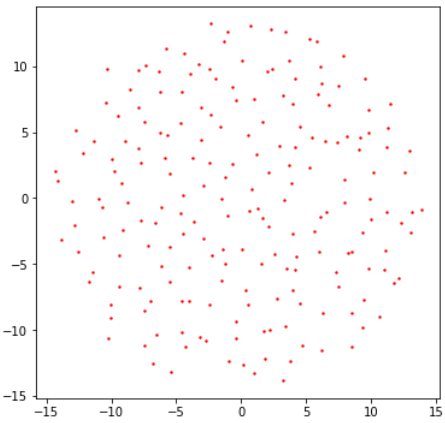


Figure 6: A realization of 200 points of the Ginibre point process with kernel (7.4.10).

However, one runs into a particular problem when simulating the truncated Ginibre process: the support of its law is the whole of $\mathbb{C}^N$. Moreover projecting onto a compact subset randomizes the number of points in the process. This method is applied only where the point process need not be in

a fixed compact set. Thus if our goal is to construct a gravitational allocation to N Ginibre points in a fixed compact set following the previous idea, an inevitable problem will occur: **" We can't control the number of points, in this fixed compact!"**. In other words, simulating N point of the Ginibre this way, we can't hope that they will fall in a space of area N or $N\pi$ and we didn't have an idea about the area of the space that they'll fall in, and having a compact we can't simulate N points of the Ginibre process that fall onto this compact.

In the article [14] the authors give an efficient way to truncate the Ginibre point process in a compact set. Their method rely on the modification of the kernel associated with the Ginibre point process. They study in depth the projection of the kernel onto a compact and its truncation to a finite rank.

7.4.1 Truncated Ginibre process on a compact subset

In this section, our aim is to specify the optimal choice of the compact subset onto which we project the truncated Ginibre point process.
We begin by studying the general projection of the truncated Ginibre process onto a centered ball of radius $R \geq 0$ which is a Determinantal point process whose law can be explicited.
By **optimal** R we mean that we hope to optimize the error that we are making by truncating the point process to $\mathbf{B}(0,R)$, as much as possible, so that we need to find the smallest radius R such that after projecting the point process into the ball of radius R the intensity of the projected point process converges to $1/\pi$, and the average number of points that fall outside this compact tends to 0.
For that end we wish to study $\mathbb{K}_R^N := P_{\mathbf{B}(0,R)}\mathbb{K}^N P_{\mathbf{B}(0,R)}$ (where $\mathbb{K}^N$ is the truncated kernel (7.4.9) and for A a compact of $\mathbb{C}$, P_A is the projection from $\mathbf{L}^2(\mathbb{C}, dl)$ onto $\mathbf{L}^2(A, dl)$ i.e. for $f \in \mathbf{L}^2(\mathbb{C}, dl)$ we have $P_A f = f1_A$), of the integral operator $\mathbb{K}$ onto $\mathbf{L}^2(\mathbf{B}(0,R), dl)$. The associated kernel is

$$\mathbb{K}_R^N(z_1, z_2) = \sum_{n=0}^{N-1} \lambda_n^R \phi_n^R(z_1)\bar{\phi}_n^R(z_2), \tag{7.4.11}$$

for z_1 , $z_2 \in \mathbf{B}(0,R)$. We denote $\mathcal{G}_R^N$ the associated point process.
$\mathcal{G}_R^N$ doesn't have N points a.s.. However, it's known that it has less than N points a.s. [16].
Optimal R:
Now, we wish to determine the optimal $R \geq 0$ onto which we project the Truncated Ginibre point process. In regards to this question, we recall that the particle density ρ_1 of the general Ginibre point process is constant, and

$$\rho(z) = \mathbb{K}(z,z) = \frac{1}{\pi},$$

for $z \in \mathbb{C}$.
However, the particle density of the Truncated Ginibre process is not constant. If we denote by ρ_n^N the n-th correlation function of $\mathcal{G}^N$, then we have

$$\rho_1^N(z) = \frac{1}{\pi} e^{-\frac{1}{2}z^2} \sum_{k=0}^{N-1} \frac{|z|^{2k}}{k!},$$

for $z \in \mathbb{C}$. Thus, we have $\int_{\mathbb{C}} \rho_1^N(z)dz = N$ as well as

$$\rho_1^N(z) \leq \frac{1}{\pi}, \ \forall z \in \mathbb{C},$$

we have also that

$$\rho_1^N(\sqrt{N}z) \xrightarrow[N\to\infty]{} \frac{1}{\pi} 1_{|z|\leq 1} ,$$

which is known as the circular law in stochastic matrix theory. Therefore, it appears that it is optimal to project onto $\mathbf{B}(0,\sqrt{N})$.
The authors of [14] have proved that, as $N \to \infty$, the average number of points falling outside of $\mathbf{B}(0,\sqrt{N})$ is of the order $\frac{1}{\sqrt{N}}$. Hence, the best strategy is to project the Truncated Ginibre point process of rank N onto $\mathbf{B}(0,\sqrt{N})$.
Simulation:
To simulate $\mathcal{G}^N_{\sqrt{N}}$, the simplest way is to simulate $\mathcal{G}^N$ and to project onto $\mathbf{B}(0,\sqrt{N})$ which is the optimal choice. (Figure 7).
Therefore when we need to simulate a truncated Ginibre on a compact set, we no longer control the number of points, i.e. there is again a random number of points in the compact set! And if we want to condition the number of point on being equal to N, there is a number of points falling outside of the ball $\mathbf{B}(0,\sqrt{N})$ which grows as $\frac{1}{\sqrt{N}}$ as N goes to infinity. Since the projection onto $\mathbf{B}(0,\sqrt{N})$ of the truncated Ginibre process takes the Determinantal form (7.4.11), one can calculate the probability of all the points falling in $\mathbf{B}(0,\sqrt{N})$.
Indeed we have

$$\mathbb{P}\big(Card(\mathcal{G}^N_{\sqrt{N}} \cap \mathbf{B}^c(0,\sqrt{N})) = 0\big) = \prod_{n=0}^{N-1} \frac{\gamma(n+1,N)}{n!},$$

and this probability tends to 0 as N tends to infinity.
Therefore, if we are required to simulate the Ginibre process on a compact conditionally on it having N points, the conditioning requires more an more computation time as N tends to infinity.

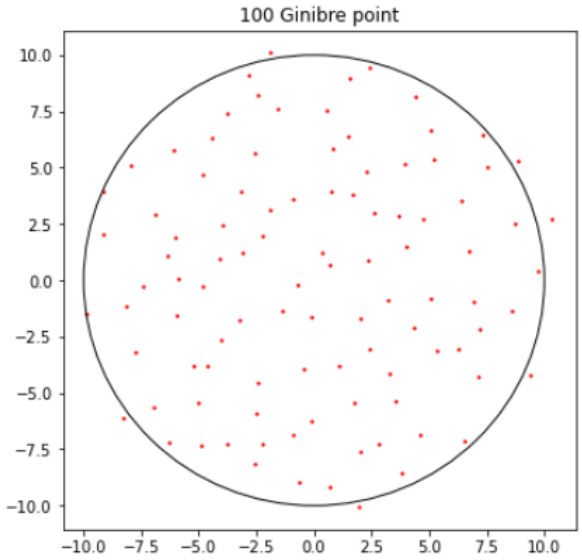


Figure 7: A realization of $\mathcal{G}^N$ and its projection onto $\mathbf{B}(0,\sqrt{N})$, with $N = 100$.

7.4.2 Truncated Ginibre point process with a fixed number of point.

As we see in the previous subsection, the problem was that if we condition on having N point in $\mathbf{B}(0,\sqrt{N})$, the conditioning requires more and more computation time as N tends to infinity. However, we are not forced to simulate the conditioning on being N points. Instead, we introduce a new kernel, as well as the associated point process by setting:

$$\mathbb{K}^N_{cond}(z_1,z_2) = \sum_{0}^{N-1} \phi^N_n(z_1)\bar{\phi}^N_n(z_2), \qquad z_1,\, z_2 \in \mathbf{B}(0,R), \tag{7.4.12}$$

where, ϕ_n^N corresponds to the function ϕ_n restricted to the compact $\mathbf{B}(0,\sqrt{N})$ (after renormalization). We emphasize that this is in fact $\mathcal{G}^N_{|_{\mathbf{B}(0,\sqrt{N})}}$ conditioned on there being N points in the compact $\mathbf{B}(0,\sqrt{N})$. Moreover, the fact that we can explicit the projection kernel associated with the conditioning is what ensures the efficiency of the simulation.
Decreusefond, Flint and Vergne in there article [14], have proved that $\mathcal{G}^N_{cond}$, the associated Determinantal process with kernel $\mathbb{K}^N_{cond}$, converges to $\mathcal{G}$ weakly as N tends to infinity, and give an algorithm to simulate $\mathcal{G}^N_{cond}$.
Simulation:
In order to simulate on $\mathbf{B}(0,a)$ instead of $\mathbf{B}(0,\sqrt{N})$, we need to apply a homothetic transformation to the N points, which translates to a homothety on the eigenvectors. To sum up, the simulation algorithm of the truncated Ginibre process on a centered ball of radius $a \geq 0$ is as follows :

Algorithm 1 Simulation of truncated Ginibre process on a compact [14]

Define $\phi_k(z) = \frac{N}{\pi a^2 \gamma(k+1,N)} e^{-\frac{N}{2a^2}|z|^2} (\frac{Nz}{a^2})^k$, for $z \in \mathbf{B}(0,a)$ and $0 \leq k \leq N-1$
Define $v(z) := (\phi_0(z), \dots \phi_{N-1}(z))$, for $z \in \mathbf{B}(0,a)$.
Sample X_N from the distribution with density $p_N(x) = \frac{|v(x)|^2}{N}$, $x \in \Lambda$
Set $e_1 = v(X_N)/|v(X_N)|$
for $i = N-1 \to 1$ **do**
    Sample X_i from the distribution with density $p_i(x) = \frac{1}{i}\left[|v(x)|^2 - \sum_{j=1}^{N-i} |e_j^* v(x)|^2\right]$
    $w_i = v(X_i) - \sum_{j=1}^{N-i} (e_j^* v(X_i)) e_j$
    $e_{N-i+1} = w_i/|w_i$
end for
Return $(X_1, \dots, X_N)$

Note that γ is the lower incomplete Gamma function define as

$$\gamma(z,a) := \int_0^a e^{-t} t^{z-1} dt.$$

The resulting is a Determinantal point process of kernel (7.4.12). Its support is on the compact ball $\mathbf{B}(0,a)$ and has almost surely N points.

7.5 Transportation of Lebesgue to the Truncated Ginibre points by gradient flow

Let $\mathcal{G}^N_{|_{\mathbf{B}(0,\sqrt{N})}} = \{z_1, z_2, \dots, z_N\}$ be a realisation of N points of the truncated Ginibre point process in the complex plane $\mathbb{C}$, simulated as mentioned in the previous subsection such that $\mathcal{G}^N$ fall in $\mathbf{B}(0,\sqrt{N})$.
To simplify the notations we will denote in what follow $\mathcal{G}^N_{|_{\mathbf{B}(0,\sqrt{N})}}$ simply by $\mathcal{G}^N$, and keep in mind that we condition on being N points in the ball $\mathbf{B}(0,\sqrt{N})$.
For all $z \in \mathbb{C}$ we define

$$f(z) := \prod_{k=1}^{N} (z - z_k),$$

so that $\mathcal{G}^N = f^{-1}\{0\}$, and define the potential function u to be

$$u(z) = \log|f(z)| - \frac{1}{2}|z|^2. \tag{7.5.13}$$

For any $z \in \mathbb{C} \setminus \mathcal{G}^N$ we define the force function F by

$$F(z) = -\nabla u(z) = -\sum_{k=1}^{N} \frac{1}{\bar{z} - \bar{z}_k} + z, \tag{7.5.14}$$

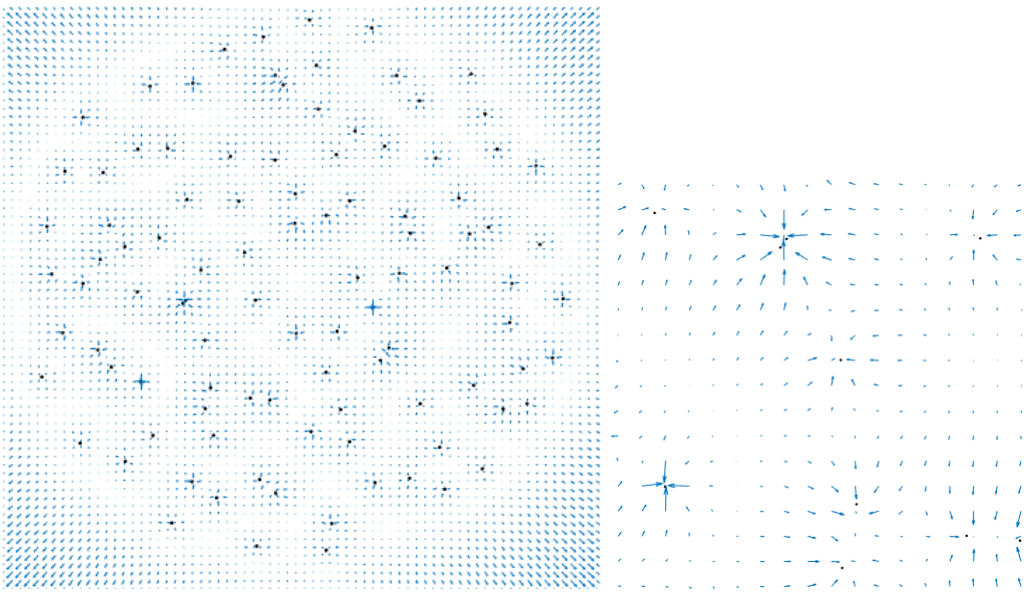


Figure 8: Quiver plot representing the Force field F (7.5.14) on the left, and a part of it on the right.

At a point z each zero z_k exerts a "gravitational force" of magnitude $\frac{1}{|z-z_k|}$ towards z_k. It's worth recalling here that **the correct analogue of the gravitational potential (equivalently the Green's function for the Laplacian) in two dimensions is** $\log|z-w|$ **while in** $\mathbb{R}^d$ **for** $d \geq 3$**, it's** $|x-y|^{-d+2}$.
Note that the force F represents the speed of a particle, rather than being proportional to its acceleration as in Newtonian gravitation.
For any $z \in \mathbb{C} \setminus \mathcal{G}^N$ consider the integral curve $Y_z(t)$ defined as solution of the differential equation

$$\frac{dY_z}{dt}(t) = -\nabla u(Y_z(t)), \qquad Y_z(0) = z, \tag{7.5.15}$$

over some maximal domain $]-\infty, \tau_z[$, where $0 < \tau_z \leq \infty$.
Visualizing the potential as a height function, we may interpret these flow lines as the trajectories of particles without inertia in a gravitational field.
We want to orient the gradient curves in the direction of decay of u (this is the reason of our choice of the minus sign in above differential equation). As we have

$$\Delta u = 2\pi \sum_{z \in \mathcal{G}^N} \delta_z - 2,$$

in distributional sense, then outside the set $\mathcal{G}^N$, the potential u is **super-harmonic**.
Therefore for "typical" initial z, the gradient curve will flow down to a star i.e. a point of $\mathcal{G}^N$.
For $z_k \in \mathcal{G}^N$, let

$$\mathcal{B}(z_k) = \{z \in \mathbb{C};\, \nabla u(z) \neq 0 \,\text{and}\, Y_z \,\text{terminates at}\, z_k\} \cup \{z_k\},$$

be the basin associated to each star z_k for $k \in 1, ..., N$.
Define:

$$\mathcal{T}(z) := \begin{cases} z_k & If \lim_{t \to \tau_z} Y_z(t) = z_k \; and \; z_k \in \mathcal{G}^N, \\ \infty & Otherwise \end{cases}$$

to be the transportation.
This construction define a transportation by gradient flow of the Lebesgue to the Truncated Ginibre point process. It's similar to the gravitational allocation by the concept, but $\mathrm{Vol}(\mathcal{T}^{-1}(\infty)) = 0$ is not be preserve. We can see it, as gravitational allocation on a finite space.

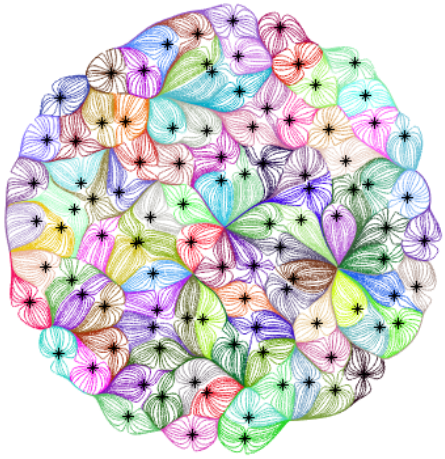


Figure 9: The basins $\mathcal{B}(z_k)$, for $z_k \in \mathcal{G}^N$, due to the transportation $\mathcal{T}$.

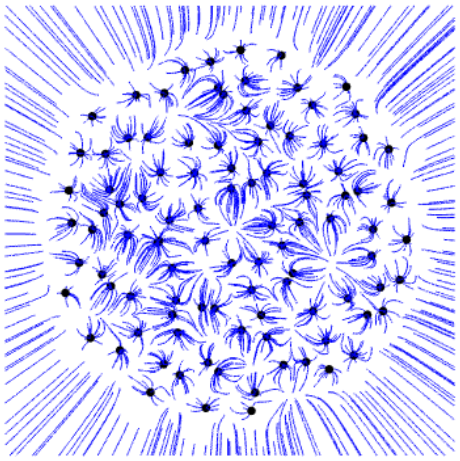


Figure 10: Trajectories of points of the space under the gradient flow.

The proof of the conditional fairness of the allocation in the previous section 7.3.1 doesn't rely on the number of points in the space. In fact, when the potential takes a particular form that mimics the gravitational potential of Newtonian mechanics, all the stars will allocate basins of equal area. Furthermore, if we define

$$u_c(z) = \log|f(z)| - \frac{1}{2} \times c \times |z|^2,$$

to be the potential function, then each star will have a basin of area π/c, *conditional on having each basin bounded by finitely many smooth gradient curves.* In our case, each basin will have a volume equal to π.
In the previous section, we give a proof of a conditional fairness of the allocation. In what follow we give a formal proof that each basin has an area equal to π, without any previous assumption.

Proposition 7.5.1. "Fairness of the allocation"
Each basin of attraction due to the allocation rule (7.5) has an area equal to π.

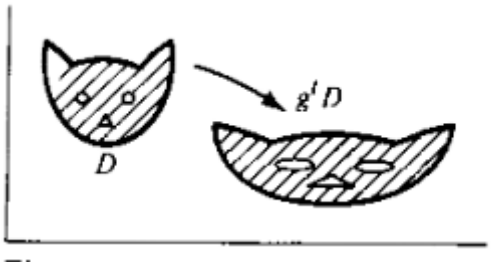
Theorem 7.5.1. "Liouville's theorem" [12]
Suppose we are given a system of ordinary differential equations

$$\dot{x} = f(x)$$

where $x = (x_1, ..., x_n)$, whose solution may be extended to the whole time axis.
Let $\{g^t\}$ be the corresponding group of transformations:

$$g^t(x) = x + f(x)t + O(t^2), \ (t \longrightarrow 0).$$

Let $D(0)$ be a region in x- space and $v(0)$ its volume; $v(t)$ is the volume of $D(t)$, and $D(t) = g^t D(0)$.



If $\mathrm{div} f \equiv 0$, then g^t preserves volume i.e. $v(t) = v(0)$.

Lemma 7.5.1.

$$(dv/dt)_{|t=t_0} = \int_{D(t_0)} \mathrm{div}\ f dx,$$

with $dx = dx_1 ... dx_n$.

Proof. of proposition 7.5.1
We will apply lemma 7.5.1 to the differential equation

$$\frac{dY_z(t)}{dt} = F(Y_z(t)), \qquad Y_z(0) = z.$$

In our case, the gravitational flow represents the transformation g^t, and if we take a region $D(0)$ of the space, then $D(t)$ is the volume obtained from $D(0)$ at time t, i.e. after the displacement of the points of $D(0)$, by the gravitational flow.
For $z_k \in \mathcal{G}^N$, let $t \geq 0$ and

$$D_t = \{z \in \mathcal{B}(z_k) \text{ s.t. } \tau_z > t\}, \text{ and } V(t) = \mathrm{Volume}(D_t).$$

Write $D_{0,\epsilon} = D_0 \setminus D_\epsilon$.
Let ϕ^t denote the gravitational flow for time t, thus ϕ^t is a bijection from D_t to D_{s-t} with inverse ϕ^{-s}.
Recall that $\phi^{-s}(z) = Y_z(-s)$ is defined for all $z \in \mathbb{C}$ and $s \in (0, \infty)$, and that $\phi^0(z) = Y_z(0) = z$.
The idea is to give two estimations of $V(t) - V(0)$, and deduce by comparing them that $V(0) = \pi$.
Let Ω be an open set of D_t with compact closer in $\mathbb{C} \setminus \mathcal{G}^N$.
As $F(z) = -\nabla u(z)$, so

$$\mathrm{div} F(z) = -\Delta u(z) = 2 - 2\pi \sum_{k=1}^{N} \delta_{z_k}(z).$$

So for $z \in \Omega$, we have

$$\mathrm{div}(F(z)) = 2.$$

By lemma 7.5.1, we obtain for $0 \leq s < t$ that :

$$\frac{d}{ds} V(\phi_s(\Omega)) = \int_{\phi_s(\Omega)} \mathrm{div}(F) dl = \int_{\phi_s(\Omega)} 2 dl = 2V(\phi_s(\Omega)).$$

So we get the differential equation

$$\frac{d}{ds} V(\phi_s(\Omega)) = 2V(\phi_s(\Omega)).$$

Solving this differential equation yields

$$V(\Omega) = e^{-2t} V(\phi_t(\Omega)).$$

Since any measurable subset of D_0 can be approximated by a set of the form $\phi_t(\Omega)$, and using Taylor expansion of exponential for $t \searrow 0$, we get

$$V_0 - V_t = V_0(1 - e^{2t}) = 2V_0 t + O(t^2). \tag{7.5.16}$$

For z in a neighborhood of z_k, we have $F(z) = \frac{z_k - z}{|z - z_k|^2} + O(1)$. So,

$$\frac{d}{dt}|Y_z(t) - z_k|^2 = 2\big\langle F(Y_z(t)), Y_z(t) - z_k \big\rangle = -2 + O(|Y_z(t) - z_k|).$$

Without the error term it would be easy to solve the differential equation explicitly, this implies by approximation that, for $\epsilon \longrightarrow 0$ we have :

$$\sup_{z \in D_0} |z - z_k|^2 \leq 2\epsilon + O(\epsilon) \text{ and } \inf_{z \notin D_0} |z - z_k|^2 \geq 2\epsilon - O(\epsilon).$$

Thus

$$\mathbf{B}(z_k, \sqrt{2\epsilon} - O(\sqrt{\epsilon})) \subseteq D_{0,\epsilon} \subseteq \mathbf{B}(z_k, \sqrt{2\epsilon} + O(\sqrt{\epsilon}))$$

which means that the area of $D_{0,\epsilon}$ is equal to $2\pi\epsilon + O(\epsilon)$. This gives

$$V_0 - V_\epsilon = \lambda_n(D_0) = 2\pi\epsilon + O(\epsilon).$$

Comparing with (7.5.16), we conclude that $V_0 = \pi$, as desired. □

Proposition 7.5.2. The basins of attraction constructed due to the transportation (7.5) are bounded. Moreover,

$$\bigcup_{k=1}^{N} \mathcal{B}(z_k) \subsetneq \mathbf{B}(0, \frac{1+\sqrt{5}}{2}\sqrt{N}).$$

Proof. To show that the basins are bounded, we'll show that there exists a radius $R_N > \sqrt{N}$, such that for any $z \in \mathbf{B}^c(0, R_N)$ we have

$$\big\langle F(z), z \big\rangle \geq 0.$$

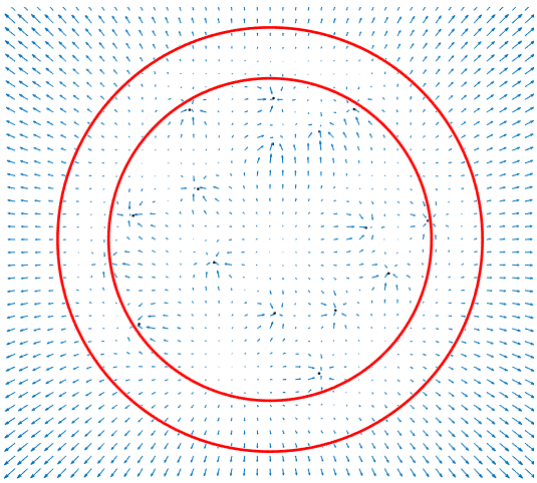
Let $R_N = \epsilon\sqrt{N}$ with $\epsilon > 1$, and let $z \in \partial\mathbf{B}(0, R_N)$,

$$\begin{aligned}
\big\langle F(z), z \big\rangle &= \big\langle -\sum_{k=1}^{N} \frac{z - z_k}{|z - z_k|^2} + z, z \big\rangle \\
&= -\sum_{k=1}^{N} \frac{\big\langle z - z_k, z \big\rangle}{|z - z_k|^2} + \big\langle z, z \big\rangle \\
&\geq -\sum_{k=1}^{N} \frac{|z - z_k||z|}{|z - z_k|^2} + \big\langle z, z \big\rangle \qquad \textbf{(Using Cauchy-Schwarz inequality)} \\
&= -\sum_{k=1}^{N} \frac{|z|}{|z - z_k|} + R_N^2 \\
&\geq -\sum_{k=1}^{N} \frac{R_N}{\sqrt{N}(\epsilon - 1)} + R_N^2 \\
&= \frac{\epsilon N\sqrt{N}}{\sqrt{N}(\epsilon - 1)} + N\epsilon^2 \\
&= \frac{N\epsilon}{\epsilon - 1}(\epsilon^2 - \epsilon + 1)
\end{aligned}$$

Let $x_1 = \frac{1+\sqrt{5}}{2}$, and

$$P(X) = X^2 - X + 1.$$

For any $X \geq x_1$ we have $P(X) \geq 0$.



Take $\epsilon = x_1$, we get that for any $z \in \mathbf{B}^c(0, \epsilon\sqrt{N})$, we have $\langle F(z), z \rangle > 0$. Therefore, the vectors $F(z)$ are all directed outside the ball $\mathbf{B}(0, \epsilon\sqrt{N})$, which means that, any point outside this ball is allocated to infinity.
Thus for any $k \in \{1, ..., N\}$, the basin $\mathcal{B}(z_k) \subsetneq \mathbf{B}(0, \epsilon\sqrt{N})$.
Thus,

$$\bigcup_{k=1}^{N} \mathcal{B}(z_k) \subsetneq \mathbf{B}(0, \epsilon\sqrt{N}),$$

which ends our proof. □

N.B. This proof doesn't depend on the configuration of the point process, but in the case of the Ginibre point process ϵ will be better than x_1.

7.6 Simulation of a gravitational allocation

As we already mentioned, the simulation of a gravitational allocation is two-fold. First, we will have to sample the considered point process, second simulates the gravitational allocation.

7.6.1 Sampling from the Ginibre point process

As, our goal is to simulate N points of the Ginibre point process, in the ball $\mathbf{B}(0, \sqrt{N})$, two methods arise:

1. Simulate a matrix A of i.i.d. $N \times N$ complex Gaussian entries, conditioned on there being in the compact $\mathbf{B}(0, \sqrt{N})$, (But this condition require more and more computation time when N get larger).

2. use the algorithm (1) defined in detail in section (7.4.2) to sample under the Truncated Ginibre.

So far, the algorithm (1) is the most efficient way, to get N points of the Ginibre in the compact $\mathbf{B}(0, \sqrt{N})$.

7.6.2 Simulation of the gravitational allocation

In this section, we propose an algorithm to visualize the basins of attraction due to the transportation (7.5).
To this aim first, we need to simulate a realization $\{z_1, ..., z_N\}$ of $\mathcal{G}^N$ using one of the two methods mentioned above (7.6.1), then to define our potential function u and evaluate its gradient vector, which represent the negative of the force function F.

Algorithm 2 Gravitational allocation from Lebesgue to the Truncated Ginibre

Sample N point of the Truncated Ginibre $\mathcal{G}^N = \{z_1, ..., z_N\}$.
Define $f(x) := \prod_{i=1}^{N}(x - z_i)$ the polynomial whose roots are the points of $\mathcal{G}^N$
Sort the points of $\mathcal{G}^N$ in increasing order and let $\{Z_1, ..., Z_N\}$ be $\mathcal{G}^N$ reordered.
Let $[X, Y]$ be the grid containing the compact $\mathbf{B}(0, \sqrt{N})$, on which we want the allocation to appear.
let $u(x) := \log|f(x)| - 0.5|x|^2$ be the potential function.
Evaluate u at $X + iY$.
$[Dx, Dy] \leftarrow gradient(u)$
Set $s = (sx, sy)$ a set of equispaced points of distance ϵ from each other, where ϵ is a small constant.
for $i = 1 \to N$ **do**
    $h \leftarrow streamline(X, Y, Dx, Dy, real(Z_k) + sx, imag(Z_K) + sy)$
    Plot Z_k
    Plot h
end for

The idea behind visualizing the basins is to surround each z_k by a quasi-circle of center z_k and radii ϵ small enough to use them as the starting point for the curved lines formed by the gradient vector.
To plot the curve line, use the function streamline of Matlab or streamplot from matplotlib in Python. In fact, these functions compute the integral curve (7.5.15) by numerical approximation using Euler's integration.

7.7 Gravitational allocation from Lebesgue to truncated Ginibre point process on the Disk

Let $\mathcal{G}^N = \{z_1, z_2, ..., z_N\}$ be a realisation of N points of the Truncated Ginibre ensemble in the ball $\mathbf{D}(0, \sqrt{N})$ i.e. the disk of radius $\sqrt{N}$ in the complex plane.
We define the gravitational allocation on the disk to $\mathcal{G}^N$, to be the transportation defined in the previous section (7.5), but this time due to the potential restricted to the disk as follows:
for any $z \in \mathbf{D}(0, \sqrt{N})$, we define the potential function to be,

$$u_{\mathbf{D}(0,\sqrt{N})}(z) = \log|f(z)| - \frac{1}{2}|z|^2. \tag{7.7.17}$$

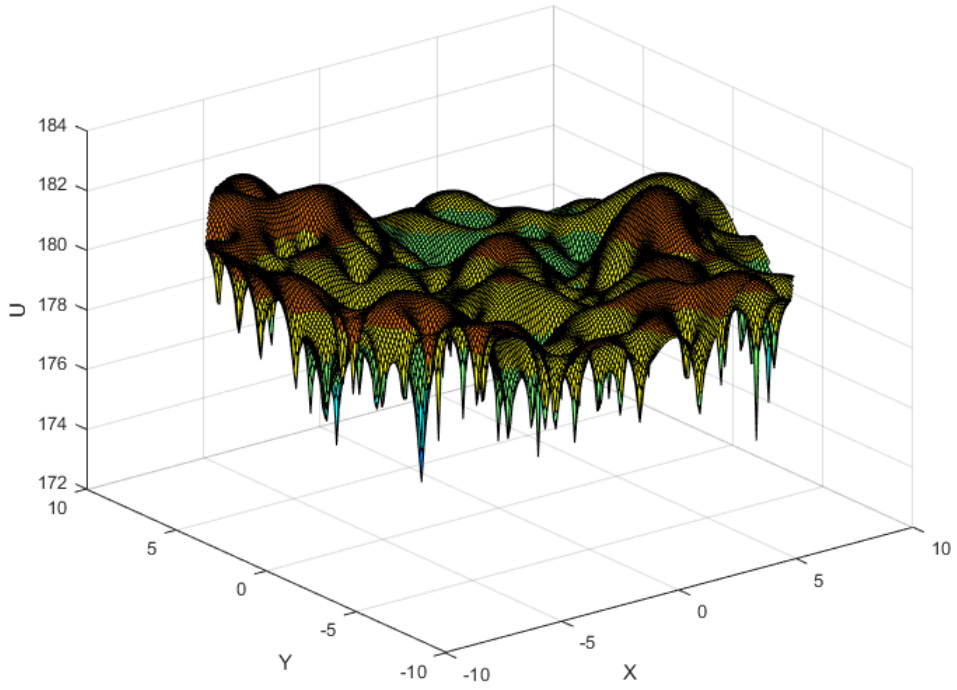


Figure 11: The potential function defined in equation (7.7.17).

In this new construction, we are sure that the basins that do not touch the boundary of the disk $\mathbf{D}(0, \sqrt{N})$ has an area equal to π, as they took enough space to complete their volume.

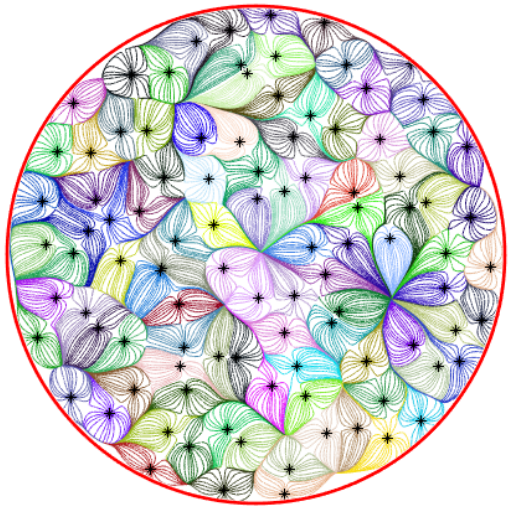


Figure 12: The basins $\mathcal{B}(z_k)$, for $z_k \in \mathcal{G}^N$, restricted to the disk of radius $\sqrt{N}$.

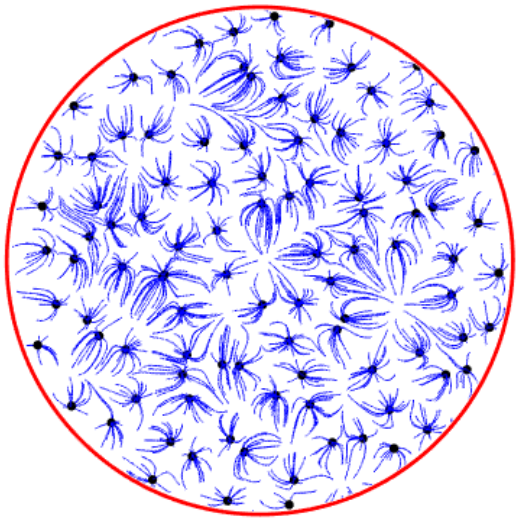


Figure 13: Trajectories of points of the disk of radius $\sqrt{N}$ under the gradient flow.

Open problems

Given a disk of area πN, suppose that we hope to divide the disk into cells of equal area, according to the gravitational allocation to N points.
Which point process is the best? (Best in the sense that $\mathrm{Vol}(\mathcal{T}^{-1}(\infty))$ is the smallest possible).
We can see (figure 14, 15, 16) that the gravitational allocation from the disk of radii 10 to 100 points of Truncated Ginibre, is better than that to 100 independent uniformly distributed points in the disk also, and it's better than that to 100 zeros of the Gaussien Entire function (G.E.F.).

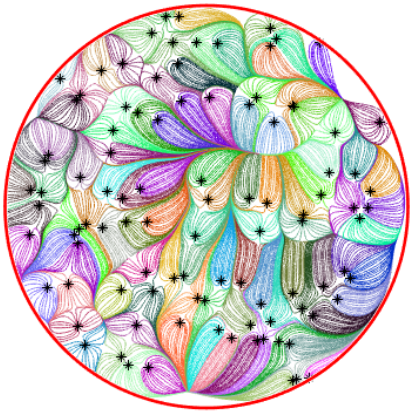


Figure 14: The allocation of the disk to 100 independent uniform points.

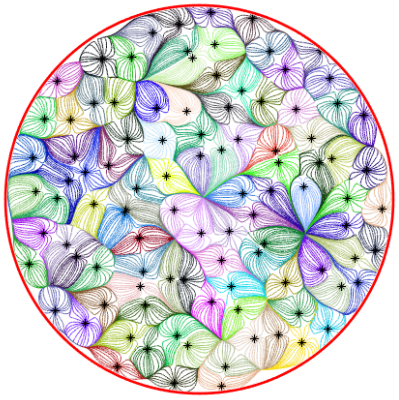


Figure 15: The allocation of the disk to 100 Truncated Ginibre points.

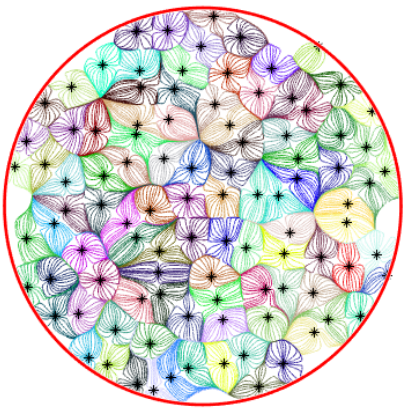


Figure 16: The allocation of the disk to 100 zeros of the Gaussien entire function.

8 Conclusion

To understand the basic ideas behind the construction of the gravitational allocation from Lebesgue to the Poison point process, we have seen all previous works under the fields of allocations [2, 6, 7, 9], and then after understanding the hidden ideas under these constructions, we tried to construct a gravitational allocation to the Ginibre point process.
To construct the new gravitational allocation, we were inspired by the gravitational allocation from Lebesgue to the zeros of the Gaussian analytic function (G.E.F.) [6, 9], since in [5], it was proved that the Ginibre point process is the set of zeros of a G.A.F. with a randomized kernel. Based on the idea of considering the target point process as the set of zeros of an analytic function, we defined the gravitational allocation to the Finite Ginibre point process, by taking the potential associated to the function whose zeros are the points of the point process. But we have encountered a technical problem which was that the intensity of the Finite Ginibre point process is not constant, so how could we control the volume of the basins in that case?
To solve this problem we have referred to the paper [14] which describes a way to truncate and project the kernel of the Ginibre point process to have N points in a ball of area πN. We proved that our construction is fair, that the basins are bounded, and that the union of the basins lies in a ball centered at the origin and of radius $\frac{1+\sqrt{5}}{2}N$. Furthermore, we propose an algorithm to visualize the basins.

Maybe, we do not have to find a 'new theory' but to build on existing theories, to combine results, to gather theories, to try to resolve problems by looking for the existence hidden solutions, since solutions may already exist but in a place where we didn't expect to find them.
So we just have to look deeply.
Each time we want to look for something new, or to resolve problems, the first step is to see all previous works under the subject that we are studying since maybe what we consider new is already studied, and what appears to be a problem is already resolved ! Since our target is to progress.

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